Convex extension of the Busemann–Hausdorff area integrand and the Plateau problem in arbitrary codimension

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This is a revised second version. In case there are any typos or mistakes with regard to contents, please do not hesitate to contact me at pistre@instmath.rwth-aachen.de.

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Zusammenfassung

Diese Masterarbeit verallgemeinert ein Result über die Minimierer des Busemann–Hausdorff Flächenfunktionals auf beliebige Kodimension. Im Jahr 2014 wurde das Plateau Problem im Finslerraum $(\mathbb{R}^3, F)$ von Overath und von der Mosel gelöst, [OvdM14]. Die vorliegende Arbeit zeigt, dass die Beweistechnik von Overath und von der Mosel erweitert werden kann, um die Existenz von flächenminimierenden Oberflächen im Finslerraum $(\mathbb{R}^n, F)$ nachzuweisen, falls die Finslermetrik reversibel ist. Um dieses Ziel zu erreichen, wird Burago and Ivanov's Arbeit [BI12] über die Konvexität der zwei-dimensionalen Busemann–Hausdorff Flächendichte ausführlich diskutiert. Abschließend wird ein Zusammenhang zwischen dem Busemann–Hausdorff Flächenintegranden und der Theorie über Cartan Integranden aufgezeigt.

Abstract

This thesis generalises a result on minimisers of the Busemann–Hausdorff area functional to arbitrary codimension. Recently, Overath and von der Mosel solved the Plateau problem for three-dimensional Finsler space $(\mathbb{R}^3, F)$, [OvdM14]. The present work shows that their proof technique can be extended to show the existence of area minimising surfaces in n-dimensional Finsler space $(\mathbb{R}^n, F)$ for reversible Finsler metrics. To achieve this goal, this thesis extensively discusses Burago and Ivanov's work [BI12] on the convexity of the two-dimensional Busemann–Hausdorff area density. Finally, a connection of the Busemann–Hausdorff area integrand and the well-investigated theory of Cartan integrands is illustrated.

About the second version

This is a revised second version, dating 31.12.2016. Apart from typing errors, the result of Lemma 1.1.14 was slightly improved. In case there are any further typos or mistakes with regard to contents, please do not hesitate to contact me at pistre@instmath.rwth-aachen.de.

Contents

Introduction

In this thesis we investigate the Plateau problem in n-dimensional Finsler space for reversible Finsler metrics F.

The Plateau problem (named in honour of the Belgian physicist J.A.F. Plateau (1801–1883)) is one example of a boundary value problem for minimal surfaces. In 1873, Plateau conducted a number of soap film experiments during which he noted that every single closed wire, however complicated its geometric form may be, bounds at least one soap film. By Johan Bernoulli's principle of virtual work, soap films in stable equilibrium correspond to surfaces of minimal surface area. Mathematically, a closed wire can be modelled by a closed rectifiable Jordan curve. In addition, one can prove that surfaces of minimal area are minimal surfaces (surfaces whose mean curvature vanishes). The question which the Plateau problem poses is the following (see [DHKW92, Chapter 4, pp. 221-226]):

Given a closed rectifiable Jordan curve Γ, is there a minimal surface spanned by Γ?

The first question one needs to answer is how to "measure" surface area. This depends on the geometric setting one studies the surfaces in. For the purpose of this thesis, we consider surfaces immersed into a Finsler space and consequently the notion of Busemann–Hausdorff area (or *Finsler area*) introduced by Busemann in [Bus47] (see Section 2.1). Secondly, one needs to discuss the class of permitted surfaces. For example, we do not consider fractal surfaces. Then we want to solve the variational problem

$$\text{minimise } \operatorname{area}_B^F(X)$$

over a class of admissible surfaces $X \in \mathcal{C}(\Gamma)$ (which will be defined in Section 3.1).

In [OvdM14], Overath and von der Mosel showed the existence of Finsler area minimisers in codimension one and established higher regularity of solutions. However, Overath and von der Mosel considered a more general class of Finsler metrics whose "m-harmonic symmetrisation" is a Finsler metric as well. In their work they applied the theory of Cartan functionals to Finsler area. For this, assume $X : \mathcal{M} \to \mathbb{R}^n$ is a smooth immersion of a smooth m-manifold into $\mathbb{R}^n$ and

$I \in C^0(\mathbb{R}^n \times \mathbb{R}^N)$, $N = \binom{n}{m}$ is positively homogeneous in its second argument, that is,

$$I(x, ty) = tI(x, y)$$

for all $(x, y) \in \mathbb{R}^n \times \mathbb{R}^N$ and $t > 0$. Therefore, the m-form

$$i(p) = I\left(X(p), dX_p\left(\left.\frac{\partial}{\partial u^1}\right|_p\right) \wedge dX_p\left(\left.\frac{\partial}{\partial u^2}\right|_p\right) \wedge \cdots \wedge dX_p\left(\left.\frac{\partial}{\partial u^m}\right|_p\right)\right).$$

for a local coordinate chart $(u^\alpha)_{\alpha=1}^m$ on $\mathcal{M}$ is globally well-defined. Then

$$\mathcal{I}(X) = \int_{p \in \mathcal{M}} i(p),$$

is called *Cartan functional* and I is the corresponding *Cartan integrand*. A broad range of results in the theory of Cartan functionals has been established. Hildebrandt and von der Mosel, in particular, showed the existence and regularity of Cartan minimisers in a certain class of admissible surfaces ([HvdM03b, HvdM03c]). Their result holds for Cartan integrands which are *positive definite*, that is,

$$M_1|z| \leq I(x, z) \leq M_2|z|$$

for all $(x, z) \in \mathbb{R}^n \times \mathbb{GC}_m(\mathbb{R}^n)$ and *semi-elliptic*, that is,

$$I(x, tz_1 + (1-t)z_2) \leq tI(x, z_1) + (1-t)I(x, z_2)$$

for all $x \in \Omega$, $z_1, z_2 \in \mathbb{R}^N$ and $t \in [0, 1]$ (see Section 3.2).

In codimension one ($n = m + 1$), Overath and von der Mosel identified the two-dimensional Busemann–Hausdorff area integrand as a positive definite, semi-elliptic Cartan integrand so that the existence result for Cartan minimisers yields a solution to the Plateau problem in Finsler space. The crucial part for the positive definiteness (see Theorem 3.3.5) is a representation of the Busemann–Hausdorff area integrand as a spherical integral (see Theorem 3.3.2) found by Overath in [Ove13] which holds in arbitrary codimension. The semi-ellipticity of the Busemann–Hausdorff area integrand for reversible Finsler metrics has been proved by Busemann in [Bus49] – but only in codimension one.

Recently, Burago and Ivanov [BI12] established this semi-ellipticity in arbitrary codimension (see Theorem 2.2.2). Their work uses concepts from multilinear algebra, convex geometry and convex analysis. Essentially, they showed that the convexity condition for the Busemann–Hausdorff area integrand is equivalent to the existence of so-called calibrators which support the area integrand "in every direction" (see Lemma 2.2.4). Subsequently, they reformulate the notion of a calibrator into an inequality regarding the Euclidean area of centrally symmetric two-dimensional polygons on the plane (see inequality (2.2.22)). This inequality is proved with elementary results from convex geometry and convex analysis (see Theorem 2.2.7).

Therefore, in the last chapter we combine Overath and von der Mosel's proof technique with the result found by Burago and Ivanov and solve the Plateau problem in Finsler space in arbitrary codimension for a reversible Finsler metric.

The thesis is outlined as follows. Chapter 1 begins with basic definitions and results on multilinear algebra, convex and differential geometry. These results are grouped into sections according to the respective topics. In particular, we introduce in Section 1.1 the algebraic notions of the tensor product and exterior power of a vector space. Subsequently, the Plücker embedding and some properties are presented. The section concludes with a Riesz-type isomorphism that arises in an inner product space. In Section 1.2 we illustrate basic principles of convex sets and polytopes such as the polarity of polytopes and polyhedral sets and the support function of a convex set. The next subsections cover properties of the mixed volumes of convex sets and we develop a well-known explicit formula for a certain mixed volume. Finally, a maximum principle for convex functions over convex domains is presented. Section 1.3 introduces basic notions of differential geometry and serves as a reference section for the subsequent chapters.

Chapter 2 covers an extensive treatment of Burago and Ivanov's work [BI12] on the convexity of the two-dimensional Busemann–Hausdorff area density. First, we formally introduce the Busemann–Hausdorff definition of volume on a Finsler manifold. This extends to the notion of the Finsler area functional area_Ω^F of immersed m-dimensional submanifolds. Secondly, we define the convexity of an area density and present Burago and Ivanov's reformulation thereof. Using the concepts introduced in Chapter 1, we prove the central result on polygons described above (see Theorem 2.2.7).

In Chapter 3 we formulate the Plateau problem in Finsler space for arbitrary codimension (see Theorem 3.1.1). Subsequently, we give basic notions and results of Cartan functional theory. In Section 3.3 we identify the Busemann–Hausdorff area integrand as a Cartan integrand which leads to a solution of the Plateau problem.

Chapter 1

Preliminaries of multilinear algebra, convex geometry and differential geometry

In this chapter we want to give an introduction to different concepts that are needed in the subsequent chapters. The focus here lies on results of multilinear algebra and convex geometry, as they are used extensively in Burago and Ivanov's work on the convexity of the two-dimensional Busemann–Hausdorff area density [BI12]. We present their work in greater detail in the next chapter. Differential geometric concepts are stated merely for reference and we omit most of the proofs in the respective section.

1.1 Multilinear algebra

This section aims to introduce some results of multilinear algebra. We begin by algebraically defining the tensor product of a finite number of vector spaces. This leads to elementary results for the exterior power $\bigwedge^m(V)$ of a vector space. In addition, the Plücker embedding arises, relating m-dimensional subspaces to a certain subset of the mth exterior power. By imposing an inner product structure on the vector space V, we introduce the notion of a volume form and prove a Riesz-type theorem. The results and notation in this section have mainly been borrowed from the treatise of Lee [Lee13, Chapter 12]. The section on the Plücker embedding is based on the book of Harris [Har95].

Some words to clarify notation. The set $\mathbb{N}$ is the set of natural numbers $\{1, 2, 3, \dots\}$ and $\mathbb{R}$ is the set of real numbers. All vector spaces are to be considered over the field of real numbers. In some parts, we commit the mild sin of identifying a linear map with its matrix representation with respect to a basis. In addition, we usually write Lv instead of $L(v)$ for the image of a vector v under a linear map L. A matrix $A \in \mathbb{R}^{m\times n}$ is represented as $(A^i_j) = (A^i_j)_{i,j} = (A^i_j)_{i=1,\dots,m,j=1,\dots,n}$ or $(A^i_j)^m_{i,j=1}$ if $m = n$. Therein i is the row and j the column index.

1.1.1 The tensor product

Suppose $V_1, V_2, \ldots, V_m$ are finite-dimensional $\mathbb{F}$-vector spaces of dimensions $n_1, n_2, \ldots, n_m$, respectively. Recall that the cartesian product $V_1 \times V_2 \times \cdots \times V_m$ turns into a vector space if it is endowed with component-wise addition and scalar multiplication given by $a(v_1, v_2, \ldots, v_m) = (av_1, av_2, \ldots, av_m)$. The cartesian product is of dimension $n_1 + n_2 + \ldots + n_m$. Our aim is to construct a new vector space $V_1 \otimes V_2 \otimes \cdots \otimes V_m$ of (finite) dimension $n_1 \cdot n_2 \cdot \ldots \cdot n_m$ which consists of linear combinations of objects of the form $v_1 \otimes v_2 \otimes \cdots \otimes v_m$ where $v_i \in V_i$ and in such a manner that $v_1 \otimes v_2 \otimes \cdots \otimes v_m$ depends linearly on each v_i separately (as opposed to the cartesian product where the scalar multiplication is not homogeneous in each entry separately). A natural algebraic way to achieve this goal is to construct this new vector space as a certain quotient vector space. We begin by forming the free vector space with basis $V_1 \times V_2 \times \cdots \times V_m$, that is, define the set of formal sums

$$\mathcal{F}(N) := \left\{ \sum_{n \in N} a_n n \;\middle|\; a_n \in K, a_n = 0 \text{ for all but finitely many } n \in N \right\}$$

where $N := V_1 \times V_2 \times \cdots \times V_m$. We can define an addition and a scalar multiplication on $\mathcal{F}(N)$ by

$$\begin{aligned} \left(\sum_{n \in N} a_n n \right) + \left(\sum_{n \in N} b_n n \right) &:= \sum_{n \in N} (a_n + b_n)\, n, \\ \alpha \left(\sum_{n \in N} b_n n \right) &:= \sum_{n \in N} (\alpha a_n)\, n. \end{aligned}$$

This turns $\mathcal{F}(N)$ into a vector space over the base field $\mathbb{F}$ of the vector spaces V_i. Take note that by definition *any* element of N is a basis element. In fact, the free vector space above is of dimension $\prod_{i=1}^m \#\mathbb{F} \cdot \dim(V_i) = (\#\mathbb{F})^m \cdot \prod_{i=1}^m \dim(V_i)$. Here $\#S$ denotes the cardinality of a set S. Therefore, when $\mathbb{F} = \mathbb{R}$ this space is far too large for our intended purpose of creating a vector space of finite dimension $n_1 \cdot n_2 \cdot \ldots \cdot n_m$. For this reason, we proceed by factoring out the linearity relations we are looking for. Consider the subspace U of $\mathcal{F}(N)$ given by

$$\operatorname{span} \big\{ (v_1, \ldots, av_i + w_i, \ldots, v_m) - a(v_1, \ldots, v_i, \ldots, v_m) - (v_1, \ldots, w_i, \ldots, v_m) \big| a \in \mathbb{R}, v_i, w_i \in V_i \big\}.$$

The ***tensor product of*** $\boldsymbol{V_1, V_2, \ldots, V_m}$, denoted as $V_1 \otimes V_2 \otimes \cdots \otimes V_m$, is defined as the quotient space

$$V_1 \otimes V_2 \otimes \cdots \otimes V_m := \mathcal{F}(V_1 \times V_2 \times \cdots \times V_m)/U.$$

The equivalence class of an element $(v_1, v_2, \dots, v_m)$ in $V_1 \otimes V_2 \otimes \cdots \otimes V_m$ will be denoted by

$$v_1 \otimes v_2 \otimes \cdots \otimes v_m := (v_1, v_2, \dots, v_m) + U.$$

and is called the ***tensor product of*** $\boldsymbol{v_1, v_2, \dots, v_m}$. By construction, the tensor product satisfies our linearity requirements

$$v_1 \otimes \cdots \otimes (av_i + w_i) \otimes \cdots \otimes v_m = a(v_1 \otimes \cdots \otimes v_i \otimes \cdots \otimes v_m) + v_1 \otimes \cdots \otimes w_i \otimes \cdots \otimes v_m$$

and any element of $V_1 \otimes V_2 \otimes \cdots \otimes V_m$ can be written as a linear combination of elements of the form $v_1 \otimes v_2 \otimes \cdots \otimes v_m$ where $v_i \in V_i$. As a real quotient space the tensor product is itself a real vector space (see e.g. [DF04, Theorem 7, p. 412]).

A map $F\colon V_1 \times V_2 \times \cdots \times V_m \to X$ into a vector space X is called ***multilinear*** if it is linear as a function of each entry separately, that is, if for each $i = 1, \dots, m$

$$F(v_1, \dots, av_i + w_i, \dots, v_m) = aF(v_1, \dots, v_i, \dots, v_m) + F(v_1, \dots, w_i, \dots, v_m).$$

The tensor product satisfies the following universal property. A proof can be found in [Lee13, Proposition 12.7, p. 309] but it will be omitted here.

Proposition 1.1.1 (Universal Property of the Tensor Product Space)
Let $V_1, V_2, \dots, V_m$ be finite-dimensional real vector spaces. If $A\colon V_1 \times V_2 \times \cdots \times V_m \to X$ is a multilinear map into a vector space X, then there is a unique linear *map $\widetilde{A}\colon V_1 \otimes V_2 \otimes \cdots \otimes V_m \to X$ such that $A = \widetilde{A} \circ \pi$, where π is the projection map given by $\pi(v_1, v_2, \dots, v_m) := v_1 \otimes v_2 \otimes \cdots \otimes v_m$. That is, the following diagram commutes:*

$$\begin{tikzcd}
V_1 \times V_2 \times \cdots \times V_m \arrow[r, "A"] \arrow[d, "\pi"'] & X \\
V_1 \otimes V_2 \otimes \cdots \otimes V_m \arrow[ur, dashed, "\widetilde{A}"'] &
\end{tikzcd}$$

The strength of the universal property and the reason for its name lie in the fact that it uniquely determines the tensor product up to unique isomorphism. Some texts also define a tensor product by its universal property and subsequently use the construction shown above to prove the existence of a tensor product (which by the following uniqueness result can then be called *the* tensor product).

Proposition 1.1.2

Let $V_1, V_2, \dots, V_m$ be finite-dimensional real vector spaces and suppose $\pi' \colon V_1 \times V_2 \times \ldots \times V_m \to Z$ is a multilinear map into a vector space Z with the following universal property.

For any multilinear map $B \colon V_1 \times V_2 \times \ldots \times V_m \to Y$ there is a unique linear *map $\widetilde{B} \colon Z \to Y$ such that $B = \widetilde{B} \circ \pi'$. That is, the following diagram commutes:*

$$\begin{tikzcd} V_1 \times V_2 \times \cdots \times V_m \arrow[r, "B"] \arrow[d, "\pi'"'] & Y \\ Z \arrow[ur, dashed, "\widetilde{B}"'] & \end{tikzcd}$$

Then there is a unique isomorphism $\Phi \colon V_1 \otimes V_2 \otimes \cdots \otimes V_m \to Z$ such that $\pi' = \Phi \circ \pi$ where $\pi \colon V_1 \times V_2 \times \cdots \times V_m \to V_1 \otimes V_2 \otimes \cdots \otimes V_m$ is the canonical projection.

Remark:

The preceding proposition shows that the details of the construction of the tensor product space are irrelevant as long as the resulting space – in the notation of Proposition 1.1.2 this is Z – satisfies the universal property.

Proof of Proposition 1.1.2: The proof will be given by diagram chasing. First, we use Proposition 1.1.1 for $X = Z$ and $A = \pi'$. This gives a unique linear map $\widetilde{A} \colon V_1 \otimes V_2 \otimes \cdots \otimes V_m \to Z$ such that $\widetilde{A} \circ \pi = \pi'$. Similarly, the universal property for Z from the hypothesis applied to the vector space $Y = V_1 \otimes V_2 \otimes \cdots \otimes V_m$ and the multilinear map $B = \pi$ yields a unique linear map $\widetilde{B} \colon Z \to V_1 \otimes V_2 \otimes \cdots \otimes V_m$ such that $\widetilde{B} \circ \pi' = \pi$. In turn, the linear composite map $C := \widetilde{A} \circ \widetilde{B} \colon Z \to Z$ satisfies $C \circ \pi' = \widetilde{A} \circ \widetilde{B} \circ \pi' = \widetilde{A} \circ \pi = \pi'$. If we reapply the universal property for Z from the hypothesis for $Y = Z$ and $B = \pi'$ we see that C is the *unique* linear map such that $C \circ \pi' = \pi'$. Of course, the identity map $\mathrm{Id}_Z \colon Z \to Z$ is also such a map and so by uniqueness we find that $C = \mathrm{Id}_Z$. Analogously (using Proposition 1.1.1 again for $X = V_1 \otimes V_2 \otimes \cdots \otimes V_m$ and $A = \pi$), one obtains $\widetilde{B} \circ \widetilde{A} = \mathrm{Id}_{V_1 \otimes V_2 \otimes \cdots \otimes V_m}$. Therefore, $\widetilde{A}$ and $\widetilde{B}$ are inverse to each other and setting $\Phi := \widetilde{A}$ proves the claim. Since $\widetilde{A}$ is unique by the universal property, so is Φ. $\square$

To establish the dimension of the tensor product $V_1 \otimes V_2 \otimes \cdots \otimes V_m$ we need the following result which involves a choice of basis for each component vector space V_i. In fact, in the above construction the vector spaces need not be finite-dimensional. The following will be the first explicit reference to a finite basis. We will omit the proof again because it only involves standard arguments from linear algebra and usage of the universal property. Essentially, we can form a basis for the tensor product space by taking all possible tensor products of basis vectors of the component spaces.

Proposition 1.1.3 ([Lee13, Proposition 12.8, p. 309])
Let $V_1, V_2, \dots, V_m$ be finite-dimensional real vector spaces of dimensions $n_1, n_2, \dots, n_m$ respectively. Suppose $\left\{e_1^{(j)}, e_2^{(j)}, \dots, e_{n_j}^{(j)}\right\}$ is a basis for V_j for each $j = 1, \dots, m$. Then the set

$$\mathcal{C} := \left\{ e_{i_1}^{(1)} \otimes e_{i_2}^{(2)} \otimes \cdots \otimes e_{i_m}^{(m)} \,\middle|\, 1 \leq i_j \leq n_j, j = 1, \dots, m \right\}$$

is a basis for $V_1 \otimes V_2 \otimes \cdots \otimes V_m$. Therefore, the dimension of $V_1 \otimes V_2 \otimes \cdots \otimes V_m$ is

$$\dim\left(V_1 \otimes V_2 \otimes \cdots \otimes V_m\right) = n_1 n_2 \cdots n_m.$$

Let us denote the set of multilinear functions $F\colon V_1 \times V_2 \times \cdots \times V_m \to \mathbb{R}$ by $L(V_1, V_2, \dots, V_m; \mathbb{R})$. It turns into a vector space if endowed with the usual pointwise addition and scalar multiplication

$$\begin{aligned}
(F + F')(v_1, \dots, v_i, \dots, v_m) &:= F(v_1, \dots, v_i, \dots, v_m) + F'(v_1, \dots, v_i, \dots, v_m) \\
(aF)(v_1, \dots, v_i, \dots, v_m) &:= a\left(F(v_1, \dots, v_i, \dots, v_m)\right).
\end{aligned}$$

The ***dual space*** of a vector space V consists of all linear functionals $\omega\colon V \to \mathbb{R}$ and will be denoted by V^*. Its elements are called ***linear forms*** or ***1-forms***. When dealing with indexed sets of elements of V and V^* we will use lower indices to represent vectors and upper indices to represent linear forms. Let $\{e_1, e_2, \dots, e_n\}$ be a basis for V. The set $\left\{\varepsilon^1, \varepsilon^2, \dots, \varepsilon^n\right\} \subset V^*$ defined by $\varepsilon^i(e_j) := \delta^i_j$ forms a basis for V^* and is called the ***dual basis*** to the initial basis for V. Here δ^i_j is the Kronecker delta, that is, $\delta^i_j = 1$ if $i = j$ and 0 otherwise. To avoid cluttering of indices we will occasionally denote the dual basis by $\{e_1^*, e_2^*, \dots, e_n^*\}$ instead.
For $\omega^i \in V_i^*, i = 1, \dots, m$ define a map $\omega^1 \otimes \omega^2 \otimes \cdots \otimes \omega^m\colon V_1 \times V_2 \times \cdots \times V_m \to \mathbb{R}$ by

$$(\omega^1 \otimes \omega^2 \otimes \cdots \otimes \omega^m)(v_1, v_2, \dots, v_m) := \omega^1(v_1)\omega^2(v_2) \cdots \omega^m(v_m). \tag{1.1.1}$$

We call this map the ***concrete tensor product of*** $\omega^1, \omega^2, \dots, \omega^m$. Note that the concrete tensor product is well-defined. This function is multilinear because $\mathbb{R}$ is a field, that is,

$$\omega^1 \otimes \omega^2 \otimes \cdots \otimes \omega^m \in L(V_1, V_2, \dots, V_m; \mathbb{R}).$$

In fact, the space of multilinear functions is spanned be such elements.

Lemma 1.1.4 ([Lee13, Proposition 12.4, p. 306])
Let $V_1, V_2, \ldots, V_m$ be finite-dimensional real vector spaces of dimensions $n_1, n_2, \ldots, n_m$, respectively. Suppose $\left\{e_1^{(j)}, e_2^{(j)}, \ldots, e_{n_j}^{(j)}\right\}$ is a basis for V_j and by $\left\{\varepsilon^1_{(j)}, \varepsilon^2_{(j)}, \ldots, \varepsilon^{n_j}_{(j)}\right\}$ denote the corresponding dual basis for V_j^* for each $j = 1, \ldots, m$. Then the set

$$\mathcal{C}' := \left\{\varepsilon^{i_1}_{(1)} \otimes \varepsilon^{i_2}_{(2)} \otimes \cdots \otimes \varepsilon^{i_m}_{(m)} \,\middle|\, 1 \le i_j \le n_j, j = 1, \ldots, m\right\}$$

is a basis for $L(V_1, V_2, \ldots, V_m; \mathbb{R})$.

Thus, we conclude that a multilinear function $F \in L(V_1, V_2, \ldots, V_m; \mathbb{R})$ is uniquely determined by its values $F_{i_1 i_2 \ldots i_m} := F\left(e^{(1)}_{i_1}, e^{(2)}_{i_2}, \ldots, e^{(m)}_{i_m}\right)$ on a basis – analogously to linear functions. By Lemma 1.1.4 and Proposition 1.1.3 both vector spaces $L(V_1, V_2, \ldots, V_m; \mathbb{R})$ and $V_1 \otimes V_2 \otimes \cdots \otimes V_m$ are of the same dimension and therefore isomorphic. Naturally, the question arises whether there is an even stronger connection between these spaces that respects the algebraic structure, that is, the multilinearity of the elements. Indeed, the following result holds.

Lemma 1.1.5 ([Lee13, Proposition 12.10, p. 311])
If $V_1, V_2, \ldots, V_m$ are finite-dimensional vector spaces, there is a canonical isomorphism such that

$$V_1^* \otimes V_2^* \otimes \cdots \otimes V_m^* \cong L(V_1, V_2, \ldots, V_m; \mathbb{R}).$$

Proof: Define a map $\Phi\colon V_1^* \times V_2^* \times \cdots \times V_m^* \to L(V_1, V_2, \ldots, V_m; \mathbb{R})$ by

$$\Phi(\omega^1, \omega^2, \ldots, \omega^m)(v_1, v_2, \ldots, v_m) := \omega^1(v_1)\omega^2(v_2)\cdots\omega^m(v_m).$$

Because each ω^i is a linear form the expression on the right is indeed a multilinear function in the arguments $v_1, v_2, \ldots, v_m$. Further, Φ is a multilinear function in the arguments $\omega^1, \omega^2, \ldots, \omega^m$ because

$$\begin{aligned}
\Phi(\omega^1, \omega^2, \ldots, a\omega^i + \widetilde{\omega}^i, \ldots, \omega^m)(v_1, v_2, \ldots, v_m) &= \omega^1(v_1)\omega^2(v_2)\cdots(a\omega^i + \widetilde{\omega}^i)(v_i)\cdots\omega^m(v_m) \\
&= \omega^1(v_1)\omega^2(v_2)\cdots\left(a\omega^i(v_i) + \widetilde{\omega}^i(v_i)\right)\cdots\omega^m(v_m) \\
&= a\omega^1(v_1)\omega^2(v_2)\cdots\omega^i(v_i)\cdots\omega^m(v_m) \\
&\quad + \omega^1(v_1)\omega^2(v_2)\cdots\widetilde{\omega}^i(v_i)\cdots\omega^m(v_m) \\
&= a\Phi(\omega^1, \omega^2, \ldots, \omega^i, \ldots, \omega^m)(v_1, v_2, \ldots, v_m) \\
&\quad + \Phi(\omega^1, \omega^2, \ldots, \widetilde{\omega}^i, \ldots, \omega^m)(v_1, v_2, \ldots, v_m).
\end{aligned}$$

By the universal property of the tensor product (Proposition 1.1.1) Φ gives rise to a unique linear map $\widetilde{\Phi}\colon V_1^* \otimes V_2^* \otimes \cdots \otimes V_m^* \to L(V_1, V_2, \ldots, V_m; \mathbb{R})$ such that

$$\widetilde{\Phi}(\omega^1 \otimes \omega^2 \otimes \cdots \otimes \omega^m)(v_1, v_2, \ldots, v_m) = \omega^1(v_1)\omega^2(v_2)\cdots\omega^m(v_m).$$

From this property we can conclude that $\widetilde{\Phi}$ takes tensor products to concrete tensor products of linear forms as defined in (1.1.1). In addition, this implies that $\widetilde{\Phi}$ takes the basis of $V_1^* \otimes V_2^* \otimes \cdots \otimes V_m^*$ given by Proposition 1.1.3 to the basis of $L(V_1, V_2, \ldots, V_m; \mathbb{R})$ given by Lemma 1.1.4, so it is an isomorphism. □

Through this identification, the elements of $V_1^* \otimes V_2^* \otimes \cdots \otimes V_m^*$ can either be regarded as elements of the abstract tensor product space or concretely as multilinear functions – whichever is more suitable. Furthermore, each vector space V_i can be canonically identified with its bidual space V_i^{**} by the isomorphism $\Phi\colon V \to V^{**}, \Phi(v)(f) := f(v)$. If we fix a basis $\{e_1, e_2, \ldots, e_n\}$ for V and denote its dual basis by $\{e_1^*, e_2^*, \ldots, e_n^*\}$ and its bidual basis (the dual of the dual basis) by $\{e_1^{**}, e_2^{**}, \ldots, e_n^{**}\}$ then it can be seen that Φ is an isomorphism. The mapping Φ sends a basis vector e_i of V to a basis vector e_i^{**} of V^{**} because

$$\Phi(e_i)(e_j^*) = e_j^*(e_i) = \delta_i^j = e_i^{**}(e_j^*).$$

Hence, Φ is bijective. Therefore, we obtain another canonical identification for the spaces

$$V_1 \otimes V_2 \otimes \cdots \otimes V_m \cong V_1^{**} \otimes V_2^{**} \otimes \cdots \otimes V_m^{**} \cong L(V_1^*, V_2^*, \ldots, V_m^*, \mathbb{R}).$$

1.1.2 The exterior power

Henceforth, we only consider a single n-dimensional vector space V. For an integer $1 \leq m \leq n$, let us denote the m-fold tensor power $V^{\otimes m} := V \otimes \cdots \otimes V$ by $T^m(V)$. An element $\alpha \in T^m(V)$ is called an ***m*-tensor of *V* of rank *m***. We want to focus on elements of $T^m(V)$ that are alternating. An m-tensor $v_1 \otimes v_2 \otimes \cdots \otimes v_m$ of V is called ***alternating*** if $v_i = v_j$ for some $i \neq j$. Denote the subspace spanned by such alternating tensors by U'. Then we define the ***m*-th exterior power of *V*** to be $\bigwedge^m(V) := T^m(V)/U'$. The equivalence class of an element $v_1 \otimes v_2 \otimes \cdots \otimes v_m$ in $\bigwedge^m(V)$ will be denoted

$$v_1 \wedge v_2 \wedge \cdots \wedge v_m := v_1 \otimes v_2 \otimes \cdots \otimes v_m + U'.$$

The elements of $\bigwedge^m(V)$ and $\bigwedge^m(V^*)$ are called ***m*-vectors** and ***(exterior) m*-forms** respectively.

Recall that we can think of both m-vectors and m-forms as multilinear functions on V^m and $(V^*)^m$ respectively. Note that these multilinear functions are alternating (that is, they evaluate to zero on repeated arguments) by the definition of the exterior power. From this point, we will almost exclusively present properties of m-vectors. Similar results hold for m-forms by exchanging the roles of V and V^* and by using the isomorphism between V and V^{**} wherever necessary.

An m-vector is said to be ***simple*** if it can be expressed in the form $\sigma = v_1 \wedge v_2 \wedge \cdots \wedge v_m$, where $v_1, v_2, \ldots v_m \in V$ are linearly independent. The subset of $\bigwedge^m(V)$ of all simple m-vectors is called the ***Grassmannian cone*** $GC_m(V)$ – the reason for this nomenclature will be explained later. It is important to notice that not every m-vector is simple (as we will see shortly) and that the representation as $v_1 \wedge v_2 \wedge \cdots \wedge v_m$ is, in general, not unique.

The significance of the simple m-vectors is that they provide a basis for $\bigwedge^m V$. Therefore, every non-simple m-vectors can be written uniquely as a linear combination of simple ones.

Lemma 1.1.6 ([Lee13, Proposition 14.8, p. 353])
Suppose $\{v_1, v_2, \ldots, v_n\}$ is a basis for V. Then for each $1 \le m \le n$ the set

$$\mathcal{L} := \{v_{i_1} \wedge v_{i_2} \wedge \cdots \wedge v_{i_m} \,|\, 1 \le i_1 < i_2 < \ldots < i_m \le n\}$$

constitutes a basis for the exterior power $\bigwedge^m(V)$. Therefore, its dimension is

$$\dim\left(\bigwedge^m V\right) = \binom{n}{m}.$$

Let $\{v_1, v_2, \ldots, v_n\}$ be a basis for V. Define the alternation mapping $\mathrm{Alt}\colon T^m V \to \bigwedge^m(V)$ on a basis m-tensor as the signed average over all permutations of the components of this tensor, that is,

$$\mathrm{Alt}(v_{i_1} \otimes v_{i_2} \otimes \cdots \otimes v_{i_m}) := \frac{1}{m!} \sum_{\sigma \in S_m} \mathrm{sgn}\,(\sigma) v_{\sigma(i_1)} \wedge v_{\sigma(i_2)} \wedge \cdots \wedge v_{\sigma(i_m)}.$$

Given an m-vector $v \in \bigwedge^m(V)$ and an l-vector $w \in \bigwedge^l(V)$ where $1 \le m + l \le n$, we define their ***wedge product*** to be

$$v \wedge w := \frac{(m+l)!}{m!l!} \mathrm{Alt}(v \otimes w).$$

The wedge product is a mapping from $\bigwedge^m(V) \times \bigwedge^l(V)$ to $\bigwedge^{m+l}(V)$.

Lemma 1.1.7 (Basic Properties of the Exterior Power)

(i) BILINEARITY: *For $a \in \mathbb{R}$, $v \in \bigwedge^m(V)$ and $w, w' \in \bigwedge^l(V)$*

$$(aw + w') \wedge v = a(w \wedge v) + (w' \wedge v),$$
$$v \wedge (aw + w') = a(v \wedge w) + (v \wedge w').$$

(ii) ASSOCIATIVITY: *For $v \in \bigwedge^k(V), w \in \bigwedge^l(V)$ and $u \in \bigwedge^m(V)$*

$$(v \wedge w) \wedge u = v \wedge (w \wedge u).$$

(iii) ANTICOMMUTATIVITY: *Let $v_1, v_2, \ldots, v_m \in V$ and $\sigma \in S_m$ be a permutation on the integers $\{1, 2, \ldots, m\}$. Then*

$$v_{\sigma(1)} \wedge v_{\sigma(2)} \wedge \cdots \wedge v_{\sigma(m)} = \operatorname{sgn}(\sigma) v_1 \wedge v_2 \wedge \cdots \wedge v_m.$$

Furthermore, for $v \in \bigwedge^m(V)$ and $w \in \bigwedge^l(V)$ and a simple m-vector $u \in GC_m(V)$

$$v \wedge w = (-1)^{ml} w \wedge v,$$
$$u \wedge u = 0.$$

(iv) *The ordered m-tuple $(v_1, v_2, \ldots, v_m)$ is linearly independent in V if and only if*

$$v_1 \wedge v_2 \wedge \cdots \wedge v_m \neq 0.$$

(v) *Suppose W is a linear subspace of V and $\mathcal{B}_1 := \{w_1, w_2, \ldots, w_m\}$ and $\mathcal{B}_2 := \{v_1, v_2, \ldots, v_m\}$ are two bases of W. Then*

$$v_1 \wedge v_2 \wedge \cdots \wedge v_m = \lambda w_1 \wedge w_2 \wedge \cdots \wedge w_m,$$

for $\lambda = \det \left(T_i^j\right)_{i,j=1,\ldots,m} \neq 0$ where $T \colon W \to W$ is the linear map sending $\mathcal{B}_1$ to $\mathcal{B}_2$.

PROOF: We omit the proofs of the parts (i) - (iii) (see [Lee13, Proposition 14.11, p. 356]). To prove part (iv) ([Lee13, Problem 14-4 (a), p. 376]) suppose $v_1, v_2, \ldots, v_m$ are linearly independent. We can extend $\{v_1, v_2, \ldots, v_m\}$ to a basis $\{v_1, v_2, \ldots, v_n\}$ of V. The top exterior power $\bigwedge^n(V)$ is one-dimensional by means of Lemma 1.1.6 and is spanned by the non-zero n-vector

$$0 \neq v_1 \wedge v_2 \wedge \cdots \wedge v_n = (v_1 \wedge v_2 \wedge \cdots \wedge v_m) \wedge (v_{m+1} \wedge v_{m+2} \wedge \cdots \wedge v_n),$$

so $v_1 \wedge v_2 \wedge \cdots \wedge v_m \neq 0$. Conversely, suppose that $v_1, v_2, \ldots, v_m$ are linearly dependent and assume without loss of generality that $v_1 = a^2 v_2 + \ldots + a^m v_m$. Then

$$v_1 \wedge v_2 \wedge \cdots \wedge v_m = \left(a^2 v_2 + \ldots + a^m v_m\right) \wedge (v_2 \wedge \cdots \wedge v_m) = 0$$

where the last equality is due to parts (i) and (iii).

For part (v) ([Lee13, Problem 14-4 (b), p. 376]) consider the linear map $T\colon W \to W$ given by $Tw_i := v_i$ for $i = 1, \ldots, m$. In slight abuse of notation we identify the linear map T with its matrix representation. Since both m-tuples span the same vector space we can express v_i as a linear combination of the w_j, that is, $v_i = Tw_i = \sum_{j=1}^m T_i^j w_j$ where $T_i^j \in \mathbb{R}$. Then by using part (i) we find that

$$\begin{aligned} v_1 \wedge v_2 \wedge \cdots \wedge v_m &= Tw_1 \wedge Tw_2 \wedge \cdots \wedge Tw_m \\ &= \left(\sum_{j_1=1}^m T_1^{j_1} w_{j_1}\right) \wedge \left(\sum_{j_2=1}^m T_2^{j_2} w_{j_2}\right) \wedge \cdots \wedge \left(\sum_{j_m=1}^m T_m^{j_m} w_{j_m}\right) \\ &= \sum_{j_1, j_2, \ldots, j_m=1}^m \left(\prod_{l=1}^m T_l^{j_l}\right) w_{j_1} \wedge w_{j_2} \wedge \cdots \wedge w_{j_m}. \end{aligned}$$

By definition of the exterior power, any m-vector is alternating. Hence, any of the summands where any two of the indices j_l coincide can be discarded. Thus, only summands where $(j_1, j_2, \ldots, j_m)$ are a permutation of $(1, 2, \ldots, m)$ add up. By this reasoning and part (iii) we can write

$$\begin{aligned} v_1 \wedge v_2 \wedge \cdots \wedge v_m &= \sum_{\sigma \in S_m} \left(\prod_{l=1}^m T_l^{\sigma(l)}\right) w_{\sigma(1)} \wedge w_{\sigma(2)} \wedge \cdots \wedge w_{\sigma(m)} \\ &= \sum_{\sigma \in S_m} \operatorname{sgn}(\sigma) \left(\prod_{l=1}^m T_l^{\sigma(l)}\right) w_1 \wedge w_2 \wedge \cdots \wedge w_m. \end{aligned}$$

The last coefficient is the determinant of the matrix of the map T. This map is a bijection since W is spanned by both $\mathcal{B}_1$ and $\mathcal{B}_2$. Thus, the determinant is non-zero. □

Now we are equipped to show that the set $GC_m(V)$ of simple m-vectors is indeed a proper subset of $\bigwedge^m(V)$ as advertised earlier.

Example:

Let $\sigma = e_1 \wedge e_2 + e_3 \wedge e_4 \in \bigwedge^2(\mathbb{R}^4)$*, where* $e_i \in \mathbb{R}^4$ *are the standard basis vectors. Then* σ *is not a simple 2-vector. To show this, suppose* $\sigma = v \wedge w$ *for linearly independent* $v, w \in \mathbb{R}^4$*. Then* $v = \sum_{i=1}^4 a^i e_i$ *and* $w = \sum_{j=1}^4 b^j e_j$*. Using the basic properties* $(i) - (iii)$ *in Lemma 1.1.7 we find*

$$\begin{aligned} e_1 \wedge e_2 + e_3 \wedge e_4 = \sigma = v \wedge w = \left(\sum_{i=1}^4 a^i e_i\right) \wedge \left(\sum_{j=1}^4 b^j e_j\right) &= \sum_{i=1}^4 \sum_{j=1}^4 a^i b^j e_i \wedge e_j \\ = \quad & a^1 b^1 e_1 \wedge e_1 + a^1 b^2 e_1 \wedge e_2 + a^1 b^3 e_1 \wedge e_3 + a^1 b^4 e_1 \wedge e_4 \\ & + a^2 b^1 e_2 \wedge e_1 + a^2 b^2 e_2 \wedge e_2 + a^2 b^3 e_2 \wedge e_3 + a^2 b^4 e_2 \wedge e_4 \\ & + a^3 b^1 e_3 \wedge e_1 + a^3 b^2 e_3 \wedge e_2 + a^3 b^3 e_3 \wedge e_3 + a^3 b^4 e_3 \wedge e_4 \\ & + a^4 b^1 e_4 \wedge e_1 + a^4 b^2 e_4 \wedge e_2 + a^4 b^3 e_4 \wedge e_3 + a^4 b^4 e_4 \wedge e_4 \\ = \quad & (a^1 b^2 - a^2 b^1) e_1 \wedge e_2 + (a^1 b^3 - a^3 b^1) e_1 \wedge e_3 + (a^1 b^4 - a^4 b^1) e_1 \wedge e_4 \\ & + (a^2 b^3 - a^3 b^2) e_2 \wedge e_3 + (a^2 b^4 - a^4 b^2) e_2 \wedge e_4 + (a^3 b^4 - a^4 b^3) e_3 \wedge e_4. \end{aligned}$$

By equating coefficients we find for example that

$$\begin{aligned} a^1 b^2 - a^2 b^1 &= 1 \\ a^3 b^4 - a^4 b^3 &= 1 \\ a^1 b^3 - a^3 b^1 &= 0 \\ a^2 b^3 - a^3 b^2 &= 0. \end{aligned} \tag{1.1.2}$$

Assume $b^3 \neq 0$ *and substitute* $a^1 = \frac{a^3}{b^3} b^1$ *into the first equation to get*

$$\frac{a^3}{b^3} b^1 b^2 - a^2 b^1 = 1.$$

But substituting the last equation of (1.1.2) *into the left hand side of the previous identity gives the contradiction*

$$\frac{a^2}{b^3} b^1 b^3 - a^2 b^1 = a^2 b^1 - a^2 b^1 = 0 \neq 1.$$

Thus, $b^3 = 0$*. But then we know from* (1.1.2) *that* $a^3 b^2 = 0 = a^3 b^1$*. Suppose* $a^3 = 0$*. This contradicts the second equation of* (1.1.2)*. So, we must have* $b^2 = 0 = b^1$ *which contradicts the first equation of* (1.1.2) *and therefore* σ *is not simple.*

There is a useful characterisation of the exterior power of the dual space. Let us define a linear mapping $\Phi\colon \bigwedge^m(V^*) \to (\bigwedge^m(V))^*$. First, define Φ on simple m-forms and m-vectors by

$$\Phi\left(\omega^{i_1} \wedge \omega^{i_2} \wedge \cdots \wedge \omega^{i_m}\right)\left(v_{j_1} \wedge v_{j_2} \wedge \cdots \wedge v_{j_m}\right) := \det\left(\omega^{i_l}(v_{j_k})\right)_{l,k=1,\ldots,m}.$$

Then extend the above definition of Φ bilinearly to all of $\bigwedge^m(V^*)$ and $(\bigwedge^m(V))^*$.

Lemma 1.1.8

The map $\Phi\colon \bigwedge^m(V^*) \to (\bigwedge^m(V))^*$ is an isomorphism. Moreover, if $\{e_1, e_2, \ldots, e_n\}$ is a basis for V

$$\Phi(e^*_{i_1} \wedge e^*_{i_2} \wedge \cdots \wedge e^*_{i_m}) = (e_{i_1} \wedge e_{i_2} \wedge \cdots \wedge e_{i_m})^*$$

where the asterisk denotes the dual basis vector in the corresponding dual space.

Proof: By definition, Φ is a linear map. Let $\{e_1, e_2, \ldots, e_n\}$ be a basis for V. Then $\{e^*_1, e^*_2, \ldots, e^*_n\}$ is the dual basis for V^* (we break with our convention of upper indices for linear forms here) and the sets $\{\, e_{i_1} \wedge e_{i_2} \wedge \ldots \wedge e_{i_m} \mid 1 \le i_1 < i_2 < \ldots < i_m \le n \,\}$ and $\mathcal{B} := \{\, e^*_{i_1} \wedge e^*_{i_2} \wedge \cdots \wedge e^*_{i_m} \mid 1 \le i_1 < i_2 < \ldots < i_m \le n \,\}$ are bases for $\bigwedge^m(V)$ and $\bigwedge^m(V^*)$, respectively. In addition, the set $\mathcal{C} := \{\, (e_{i_1} \wedge e_{i_2} \wedge \cdots \wedge e_{i_m})^* \mid 1 \le i_1 < i_2 < \ldots < i_m \le n \,\}$ forms the dual basis of $(\bigwedge^m(V))^*$. We will show that Φ maps the basis $\mathcal{B}$ to the basis $\mathcal{C}$ which proves that it is an isomorphism. Let $I = (i_1, i_2, \ldots, i_m)$ and $J = (j_1, j_2, \ldots, j_m)$ be two strictly increasing multi-indices. Calculate

$$\Phi(e^*_{i_1} \wedge e^*_{i_2} \wedge \cdots \wedge e^*_{i_m})(e_{j_1} \wedge e_{j_2} \wedge \cdots \wedge e_{j_m}) = \det\left(e^*_{i_l}(e_{j_k})\right)_{l,k=1,\ldots,m} = \det\left(\delta^{i_l}_{j_k}\right)_{l,k=1,\ldots,m}.$$

If $I = J$ then the right hand side is the determinant of the identity matrix and evaluates to 1. Suppose $I \neq J$, then there is at least one $l \in \{1, \ldots, m\}$ such that $i_l \neq j_k$ for all $k \in \{1, \ldots, m\}$, that is, there is at least one row in the matrix $\left(\delta^{i_l}_{j_k}\right)_{l,k=1\ldots m}$ which consists only of zero-valued entries. Therefore, the right hand side in the latter equation evaluates to zero. This proves that $\Phi(e^*_{i_1} \wedge e^*_{i_2} \wedge \cdots \wedge e^*_{i_m}) = (e_{i_1} \wedge e_{i_2} \wedge \cdots \wedge e_{i_m})^*$ by definition of the dual basis. □

Hereafter, we will make no distinction between these two spaces and consequently let m-forms $\omega \in \bigwedge^m(V^*)$ act on m-vectors $\sigma \in \bigwedge^m(V)$.

1.1.3 The Plücker embedding

Let us investigate the implications of Lemma 1.1.7 (v) in more detail. Consider an N-dimensional vector space $\mathcal{V}$ and define an equivalence relation $\sim$ on the set $\mathcal{V} \setminus \{0\}$ by proposing that two non-zero elements $v, w \in \mathcal{V}$ are equivalent if and only if there is a non-zero number $\lambda \in \mathbb{R}$ such that $v = \lambda w$. The equivalence class $[v]_\sim$ is the one-dimensional subspace spanned by $v \in \mathcal{V}$. The quotient $\mathbb{P}(\mathcal{V}) := (\mathcal{V} \setminus \{0\}) \,/\, \sim$ is called ***projective space***. Geometrically speaking, we can view the projective space $\mathbb{P}(\mathcal{V})$ as the set of all lines in $\mathcal{V}$ that pass through the origin, that is, the set of all one-dimensional subspaces of the N-dimensional vector space $\mathcal{V}$.

Generalising the preceding idea, let us define the set $G_m(\mathcal{V})$ of all m-dimensional subspaces of an N-dimensional vector space $\mathcal{V}$ for each $1 \leq m \leq N$. This set is called the ***Grassmannian of $\mathcal{V}$***.

We introduce a mapping $\rho\colon G_m(V) \to \mathbb{P}\left(\bigwedge^m(V)\right)$. Let $W \in G_m(V)$ be an m-dimensional subspace of V and $\{w_1, w_2, \ldots, w_m\}$ a basis of W. Define $\rho(W) := [w_1 \wedge w_2 \wedge \cdots \wedge w_m]_\sim$. By means of Lemma 1.1.7 (v) this mapping is well-defined because different choices of bases for W yield the same image in the projective space $\mathbb{P}\left(\bigwedge^m(V)\right)$. The mapping ρ is known as the ***Plücker embedding***.

Proposition 1.1.9 ([Har95, p. 64])
The Plücker embedding $\rho\colon G_m(V) \to \mathbb{P}\left(\bigwedge^m(V)\right)$ is injective.

PROOF: This proof is due to [Hud07, Proposition 2.3, p. 3]. Define $\varphi\colon \mathbb{P}\left(\bigwedge^m(V)\right) \to \bigcup_{s=1}^{n} G_s(V)$ by

$$\varphi\left([w]_\sim\right) := \left\{ v \in V \,\middle|\, v \wedge w = 0 \in \bigwedge^{m+1}(V) \right\}.$$

The set $\varphi\left([w]_\sim\right)$ is a subspace of V due to the bilinearity of the wedge product (Lemma 1.1.7 (i)). Thus, the map φ is well-defined. We will show that φ is a left-inverse of ρ, that is, $\varphi \circ \rho = \mathrm{Id}_{G_m(V)}$. Let $W \in G_m(V)$ be arbitrary with basis $\{w_1, w_2, \ldots, w_m\}$, such that $\rho(W) = [w_1 \wedge w_2 \wedge \cdots \wedge w_m]_\sim$. For each $w \in W$, it is clear by Lemma 1.1.7 (iv) that $w \wedge w_1 \wedge w_2 \wedge \cdots \wedge w_m = 0$ hence, $W \subset \varphi \circ \rho(W)$. Conversely, if $v \in \varphi \circ \rho(W)$ then $v \ \wedge w_1 \wedge w_2 \wedge \cdots \wedge w_m = 0$. Extend $(w_1, w_2, \ldots, w_m)$ to a basis $\{w_1, w_2, \ldots, w_n\}$ of V and write $v = \sum_{i=1}^n a^i w_i$. The anticommutativity of the wedge product (Lemma 1.1.7 (iii)) implies

$$\begin{aligned} 0 &= \left(\sum_{i=1}^n a^i w_i\right) \wedge w_1 \wedge w_2 \wedge \cdots \wedge w_m \\ &= (-1)^m \sum_{i=m+1}^n a^i \; w_1 \wedge w_2 \wedge \cdots \wedge w_m \wedge w_i \end{aligned}$$

and because $w_1 \wedge w_2 \wedge \cdots \wedge w_m \wedge w_i$ for $i = m+1, \ldots, n$ are linearly independent $(m+1)$-vectors by means of Lemma 1.1.6, all the coefficients a^i for $i = m+1, \ldots, n$ vanish. Thus, $v = \sum_{i=1}^m a^i w_i \in W$ and $\varphi \circ \rho(W) \subset W$. This proves that ρ is injective. □

Many more results are known about the Plücker embedding. For example, the Plücker embedding is, in fact, a topological embedding, that is, a homeomorphism onto its image and hence deserves its name. For more details, we refer to the survey article [BN91]. The Plücker embedding provides an important relationship between m-dimensional subspaces and simple m-vectors.

Lemma 1.1.10 ([Hud07, Lemma 3.8, p. 7])
An element $[w]_\sim \in \mathbb{P}\left(\bigwedge^m(V)\right)$ lies in the image of the Grassmannian under the Plücker embedding if and only if w is a simple m-vector. That is, $[w]_\sim \in \operatorname{Im}(\rho)$ if and only if $w \in GC_m(V)$.
Proof: If w can be written as $w = v_1 \wedge v_2 \wedge \cdots \wedge v_m$ for linearly independent vectors $v_1, v_2, \ldots, v_m \in V$ then the subspace of V spanned by $\{v_1, v_2, \ldots, v_m\}$ is m-dimensional, hence there is a $U \in G_m(V)$ with $U = \operatorname{span}\{v_1, v_2, \ldots, v_m\}$ and $\rho(U) = [w]_\sim$. Conversely, suppose $[w]_\sim = \rho(U)$ for some $U \in G_m(V)$. Choose a basis $\{u_1, u_2, \ldots, u_m\}$ for U. Then by definition of the Plücker embedding $[w]_\sim = [u_1 \wedge u_2 \wedge \cdots \wedge u_m]_\sim$. Thereby, w can be written as $\lambda u_1 \wedge u_2 \wedge \cdots \wedge u_m$ for some non-zero scalar $\lambda \in \mathbb{R}$, and so ω is a simple m-vector. □

Generally, an algebraic cone is a subset C of a vector space such that $\lambda c \in C$ whenever $c \in C$ and $\lambda > 0$. The Plücker embedding provides us with a tool to treat the Grassmannian as a subset of the projectivisation of simple m-vectors. This is the reason why the set of simple vectors is called the Grassmannian cone $GC_m(V)$. Thompson [Tho96, p. 196] attributes this nomenclature to Busemann, Ewald and Shepard [BES63].

1.1.4 Inner products, volume forms and a Riesz-type isomorphism

Let us conclude the section on multilinear algebra by imposing an additional structure. Suppose $(V, \langle\cdot,\cdot\rangle_V)$ is an inner product space, which is a finite-dimensional vector space with a positive-definite, symmetric, bilinear mapping $\langle\cdot,\cdot\rangle_V \colon V \times V \to \mathbb{R}$, called an ***inner product***. Notice that the inner product induces a norm

$$\|\cdot\|_V \colon V \to \mathbb{R}_+, v \mapsto \langle v, v\rangle_V^{\frac{1}{2}}$$

on V. The ***dual norm***

$$\|\omega\|_{V^*} := \sup_{\|u\|_V = 1} |\omega(u)|$$

of a linear form $\omega \in V^*$ turns the dual space V^* into a normed space. By the Riesz representation theorem [Wer00, Theorem V.3.6, p. 228] the map $J \colon V \to V^*$ given by $J(v)(u) := \langle v, u\rangle_V$ for $v, u \in V$ is an isomorphism. The dual space V^* turns into an inner product space by defining

$\langle \omega, \nu \rangle_{V^*} := \left\langle J^{-1}(\omega), J^{-1}(\nu) \right\rangle_V$. Since J^{-1} is linear and $\langle \cdot, \cdot \rangle_V$ is bilinear, so is $\langle \cdot, \cdot \rangle_{V^*}$. The symmetry of this mapping follows immediately from the symmetry of $\langle \cdot, \cdot \rangle_V$. Because J^{-1} is an isomorphism and the inner product on V is positive-definite, the positive-definiteness of $\langle \cdot, \cdot \rangle_{V^*}$ follows.

With respect to this inner product on the dual space, J is an ***isometry***, which means it is an isomorphism that preserves the inner product. In fact, the dual norm as defined above is induced by this inner product because J is an isometry. This can be seen by calculating that for some $\omega \in V^*$ and the corresponding $v \in V$ with $J(v) = \omega$ one obtains

$$\|\omega\|_{V^*} = \sup_{\|u\|_V = 1} |\omega(u)| = \sup_{\|u\|_V = 1} |J(v)(u)| = \sup_{\|u\|_V = 1} |\langle v, u \rangle_V| \leq \sup_{\|u\|_V = 1} \|v\|_V \|u\|_V = \|v\|_V.$$

Further,

$$\left| w\left(\frac{v}{\|v\|_V} \right) \right| = \left| \left\langle v, \frac{v}{\|v\|_V} \right\rangle_V \right| = \|v\|_V,$$

and thus,

$$\|\omega\|_{V^*}^2 = \|v\|_V^2 = \langle v, v \rangle_V = \left\langle J^{-1}(\omega), J^{-1}(\omega) \right\rangle_V = \langle \omega, \omega \rangle_{V^*}.$$

Moreover, we can endow the exterior power $\bigwedge^m(V)$ with an inner product. Define it on simple m-vectors by

$$\langle v_1 \wedge v_2 \wedge \cdots \wedge v_m, w_1 \wedge w_2 \wedge \cdots \wedge w_m \rangle_{\bigwedge^m(V)} := \det \left(\langle v_i, w_j \rangle_V \right)_{i,j=1\ldots m}$$

where $v_i, w_i \in V$ for $i = 1, \ldots, m$ and extend bilinearly (see e.g. [Tho96, p. 192]). The symmetry then follows from the properties of the determinant and the inner product on V. The positive-definiteness is established as follows. The matrix $G := \left(\langle v_i, v_j \rangle_V \right)_{i,j=1\ldots m} \in \mathbb{R}^{m \times m}$ – the so-called Gram matrix of $\langle \cdot, \cdot \rangle_V$ – is positive semi-definite because for $0 \neq \alpha \in \mathbb{R}^m$

$$\alpha^t G \alpha = \sum_{i=1}^m \sum_{j=1}^m \alpha_i \langle v_i, v_j \rangle_V \alpha_j = \left\langle \sum_{i=1}^m \alpha_i v_i, \sum_{j=1}^m \alpha_j v_j \right\rangle_V = \left\| \sum_{i=1}^m \alpha_i v_i \right\|_V^2 \geq 0 \qquad (1.1.3)$$

where we simply renamed the index j in the last equation. Suppose $\alpha \neq 0$ is an eigenvector of G and $\lambda \in \mathbb{R}$ its eigenvalue. Then $\lambda = \|\sum_{i=1}^m \alpha_i v_i\|_V^2 / \|\alpha\|_{\mathbb{R}^m}^2 \geq 0$ by 1.1.3. Since the determinant is the product of the eigenvalues, we have $\det(G) \geq 0$. Further, $\det(G) = 0$ if and only if at least one eigenvalue equals zero. This means $\det(G) = 0$ if and only if there is $0 \neq \alpha \in \mathbb{R}^m$ such that $\sum_{i=1}^m \alpha_i v_i = 0$. Equivalently, $\det(G) = 0$ if and only if the set $\{v_1, v_2, \ldots, v_m\}$ is linearly dependent.

Due to Lemma 1.1.7 (iv) this means $\det(G) = 0$ if and only if $v_1 \wedge v_2 \wedge \cdots \wedge v_m = 0$. Hence, $\langle \cdot, \cdot \rangle_{\bigwedge^m(V)}$ does indeed define an inner product.

Again the map $\mathcal{J}\colon \bigwedge^m(V) \to (\bigwedge^m(V))^*$ given by $\mathcal{J}(\sigma)(\tau) := \langle \sigma, \tau \rangle_{\bigwedge^m(V)}$ for $\sigma, \tau \in \bigwedge^m(V)$ is an isomorphism which is isometric with respect to the induced dual inner product on $(\bigwedge^m(V))^*$

Recall from Lemma 1.1.8 that $\bigwedge^m(V^*)$ and $(\bigwedge^m(V))^*$ are isomorphic through Φ. Since now there are inner products on each of these spaces at our disposal, the question arises whether Φ is also an isomorphism of inner product spaces not only vector spaces, that is, whether Φ preserves the inner product.

Let $\mathcal{I} := \{\, (i_1, i_2, \dots, i_m) \mid 1 \le i_1 < i_2 < \ldots < i_m \le n \,\}$ and $\{e_1, e_2, \dots, e_n\}$ be a basis for V. Further, let $F = \sum_{I\in\mathcal{I}} a^I e^*_{i_1} \wedge e^*_{i_2} \wedge \cdots \wedge e^*_{i_m}$ and $G = \sum_{I\in\mathcal{I}} b^I e^*_{i_1} \wedge e^*_{i_2} \wedge \cdots \wedge e^*_{i_m}$ be two elements of $\bigwedge^m(V^*)$. Recall that $\Phi(e^*_{i_1} \wedge e^*_{i_2} \wedge \cdots \wedge e^*_{i_m}) = (e_{i_1} \wedge e_{i_2} \wedge \cdots \wedge e_{i_m})^*$ by Lemma 1.1.8. It follows from the definition of the induced inner product on the dual space that

$$
\begin{aligned}
\langle \Phi(F), \Phi(G) \rangle_{(\bigwedge^m(V))^*} &= \sum_{I,J\in\mathcal{I}} a^I b^J \left\langle \Phi\left(e^*_{i_1} \wedge e^*_{i_2} \wedge \cdots \wedge e^*_{i_m}\right), \Phi\left(e^*_{j_1} \wedge e^*_{j_2} \wedge \cdots \wedge e^*_{j_m}\right) \right\rangle_{(\bigwedge^m(V))^*} \\
&= \sum_{I,J\in\mathcal{I}} a^I b^J \left\langle (e_{i_1} \wedge e_{i_2} \wedge \cdots \wedge e_{i_m})^*, (e_{j_1} \wedge e_{j_2} \wedge \cdots \wedge e_{j_m})^* \right\rangle_{(\bigwedge^m(V))^*} \\
&= \sum_{I,J\in\mathcal{I}} a^I b^J \left\langle e_{i_1} \wedge e_{i_2} \wedge \cdots \wedge e_{i_m}, e_{j_1} \wedge e_{j_2} \wedge \cdots \wedge e_{j_m} \right\rangle_{\bigwedge^m(V)} \\
&= \sum_{I,J\in\mathcal{I}} a^I b^J \det\left(\langle e_{i_l}, e_{j_k} \rangle_V\right)_{l,k=1,\dots,m} \\
&= \sum_{I,J\in\mathcal{I}} a^I b^J \det\left(\langle e^*_{i_l}, e^*_{j_k} \rangle_{V^*}\right)_{l,k=1,\dots,m} \\
&= \sum_{I,J\in\mathcal{I}} a^I b^J \left\langle e^*_{i_1} \wedge e^*_{i_2} \wedge \cdots \wedge e^*_{i_m}, e^*_{j_1} \wedge e^*_{j_2} \wedge \cdots \wedge e^*_{j_m} \right\rangle_{\bigwedge^m(V^*)} \\
&= \langle F, G \rangle_{\bigwedge^m(V^*)}.
\end{aligned}
$$

Thus, we have proved the following result.

Lemma 1.1.11
The spaces $\left(\bigwedge^m(V^*), \langle \cdot, \cdot \rangle_{\bigwedge^m(V^*)}\right)$ *and* $\left((\bigwedge^m(V))^*, \langle \cdot, \cdot \rangle_{(\bigwedge^m(V))^*}\right)$ *are isometrically isomorphic by means of the mapping* $\Phi\colon \bigwedge^m(V^*) \to (\bigwedge^m(V))^*$ *in Lemma 1.1.8.*

Similarly to Lemma 1.1.8 we will not distinguish these two spaces and use either of the inner products and norms - whichever is more convenient.

Two vectors $v, v' \in \mathcal{V}$ of an inner product space $\mathcal{V}$ are said to be ***orthogonal*** if $\langle v, v' \rangle_{\mathcal{V}} = 0$ and ***orthonormal*** if additionally $\langle v, v \rangle_{\mathcal{V}} = 1 = \langle v', v' \rangle_{\mathcal{V}}$. Let $\{e_1, e_2, \dots, e_n\}$ be an orthonormal basis for

V. Then for two strictly increasing multi-indices I and J

$$\langle e_{i_1} \wedge e_{i_2} \wedge \cdots \wedge e_{i_m}, e_{j_1} \wedge e_{j_2} \wedge \cdots \wedge e_{j_m} \rangle_{\bigwedge^m(V)} = \det\left(\langle e_{i_l}, e_{j_k} \rangle_V\right)_{l,k=1\dots m} = \det\left(\delta^{i_l}_{j_k}\right)_{l,k=1\dots m}.$$

By the same argumentation used in the proof of Lemma 1.1.8 the right hand side is 0 if $I \neq J$ and 1 if $I = J$. The preceding calculation proves the next result.

Corollary 1.1.12

An orthonormal basis $\{e_1, e_2, \dots, e_n\}$ for V yields an orthonormal basis $\{e_{i_1} \wedge e_{i_2} \wedge \cdots \wedge e_{i_m} \mid 1 \leq i_1 < i_2 < \ldots < i_m \leq n\}$ for the exterior power $\bigwedge^m(V)$.

Corollary 1.1.13

The spaces $\left(\bigwedge^m(\mathbb{R}^n), \|\cdot\|_{\bigwedge^m(\mathbb{R}^n)}\right)$ and $\left(\mathbb{R}^N, \|\cdot\|_{\mathbb{R}^N}\right)$, where $N := \binom{n}{m}$ and $\|\cdot\|_{\mathbb{R}^N}$ is the standard Euclidean norm on $\mathbb{R}^N$, are isometrically isomorphic.

Proof: Let $\{e_1, e_2, \dots, e_n\}$ be the standard orthonormal basis for $\mathbb{R}^n$. Due to the preceding corollary the set $\mathcal{E} := \{e_{i_1} \wedge e_{i_2} \wedge \cdots \wedge e_{i_m} \mid 1 \leq i_1 < i_2 < \ldots < i_m \leq n\}$ is an orthonormal basis for $\bigwedge^m(\mathbb{R}^n)$. Consider the set $\mathcal{I} := \left\{(i_1, i_2, \dots, i_m) \mid 1 \leq i_1 < i_2 < \ldots < i_m \leq n\right\}$ of strictly increasing multi-indices. Let $\mathfrak{E} := \{\widetilde{e}_I \mid I \in \mathcal{I}\}$ denote the standard orthonormal basis for $\mathbb{R}^N$. Since both spaces are of the same dimension $\binom{n}{m}$, let $\Phi\colon \bigwedge^m(\mathbb{R}^n) \to \mathbb{R}^N$ be the isomorphism that sends the basis $\mathcal{E}$ to the basis $\mathfrak{E}$. Then for $\sigma = \sum_{I \in \mathcal{I}} a^I e_{i_1} \wedge e_{i_2} \wedge \cdots \wedge e_{i_m} \in \bigwedge^m(\mathbb{R}^n)$ we have that

$$\|\sigma\|^2_{\bigwedge^m(\mathbb{R}^n)} = \left\| \sum_{I \in \mathcal{I}} a^I e_{i_1} \wedge e_{i_2} \wedge \cdots \wedge e_{i_m} \right\|^2_{\bigwedge^m(\mathbb{R}^n)} = \sum_{I \in \mathcal{I}} (a^I)^2$$

because of the orthonormality of $\mathcal{E}$. On the other hand

$$\|\Phi(\sigma)\|^2_{\mathbb{R}^N} = \left\| \sum_{I \in \mathcal{I}} a^I \Phi(e_{i_1} \wedge e_{i_2} \wedge \cdots \wedge e_{i_m}) \right\|^2_{\mathbb{R}^N} = \left\| \sum_{I \in \mathcal{I}} a^I \widetilde{e}_I \right\|^2_{\mathbb{R}^N} = \sum_{I \in \mathcal{I}} (a^I)^2$$

which proves the claim. $\square$

Lemma 1.1.14

For any $\sigma = w_1 \wedge w_2 \wedge \cdots w_m \in GC_m(V)$ it holds that

$$\|\sigma\|_{\bigwedge^m(V)} \leq \prod_{i=1}^{m} \|w_i\|_V \tag{1.1.4}$$

with equality if and only if $w_1, w_2, \dots, w_m$ are pairwise orthogonal.

Proof: We will take advantage of the alternating property of the wedge product. By means of the Gram–Schmidt process define the vectors

$$\begin{aligned} \widetilde{w}_1 &:= w_1 & v_1 &:= \frac{\widetilde{w}_1}{\|\widetilde{w}_1\|_V} & \\ \widetilde{w}_i &:= w_i - \sum_{j=1}^{i-1} \langle w_i, v_j \rangle_V v_j & v_i &:= \frac{\widetilde{w}_i}{\|\widetilde{w}_i\|_V}, & i = 2, 3, \dots m. \end{aligned}$$

Note that the vectors $\{\widetilde{w}_1, \widetilde{w}_2, \dots, \widetilde{w}_m\}$ and $\{v_1, v_2, \dots, v_m\}$ are pairwise orthogonal respectively. By definition of the vectors $\widetilde{w}_i$ and v_i and the multilinearity of the wedge product we see that

$$\begin{aligned} \widetilde{w}_1 \wedge \widetilde{w}_2 \wedge \cdots \wedge \widetilde{w}_m &= \widetilde{w}_1 \wedge \widetilde{w}_2 \wedge \cdots \wedge \widetilde{w}_{m-1} \wedge \left(w_m - \sum_{j=1}^{m-1} \langle w_i, v_j \rangle_V v_j \right) \\ &= \widetilde{w}_1 \wedge \widetilde{w}_2 \wedge \cdots \wedge \widetilde{w}_{m-1} \wedge w_m - \sum_{j=1}^{m-1} \langle w_i, v_j \rangle_V \widetilde{w}_1 \wedge \widetilde{w}_2 \wedge \cdots \wedge \widetilde{w}_{m-1} \wedge \frac{\widetilde{w}_j}{\|\widetilde{w}_j\|_V} \\ &= \widetilde{w}_1 \wedge \widetilde{w}_2 \wedge \cdots \wedge \widetilde{w}_{m-1} \wedge w_m \end{aligned}$$

where we used Lemma 1.1.7 (v) in the last step. Proceeding inductively, we have shown that

$$\widetilde{w}_1 \wedge \widetilde{w}_2 \wedge \cdots \wedge \widetilde{w}_m = w_1 \wedge w_2 \wedge \cdots \wedge w_m.$$

Thus, it follows that

$$w_1 \wedge w_2 \wedge \cdots \wedge w_m = \widetilde{w}_1 \wedge \widetilde{w}_2 \wedge \cdots \wedge \widetilde{w}_m = \left(\prod_{i=1}^{m} \|\widetilde{w}_i\|_V \right) v_1 \wedge v_2 \wedge \cdots \wedge v_m.$$

Use the definition of the induced norm to get

$$\begin{aligned} \|\widetilde{w}_i\|_V^2 = \langle \widetilde{w}_i, \widetilde{w}_i \rangle_V &= \left\langle w_i - \sum_{j=1}^{i-1} \langle w_i, v_j \rangle_V v_j, w_i - \sum_{k=1}^{i-1} \langle w_i, v_k \rangle_V v_k \right\rangle_V \\ &= \langle w_i, w_i \rangle_V - 2 \sum_{j=1}^{i-1} \langle w_i, v_j \rangle_V^2 + \sum_{j=1}^{i-1} \sum_{k=1}^{i-1} \langle w_i, v_j \rangle_V \langle w_i, v_k \rangle_V \underbrace{\langle v_j, v_k \rangle_V}_{=\delta_{jk}} \qquad (1.1.5) \\ &= \langle w_i, w_i \rangle_V - \sum_{j=1}^{i-1} \langle w_i, v_j \rangle_V^2 \leq \langle w_i, w_i \rangle_V = \|w_i\|_V^2. \end{aligned}$$

Subsequently, we find that

$$\begin{aligned}
\|\sigma\|_{\bigwedge^m(V)} &= \|w_1 \wedge w_2 \wedge \cdots \wedge w_m\|_{\bigwedge^m(V)} \\
&= \left(\prod_{i=1}^m \|\widetilde{w}_i\|_V\right) \|v_1 \wedge v_2 \wedge \cdots \wedge v_m\|_{\bigwedge^m(V)} \\
&\leq \left(\prod_{i=1}^m \|w_i\|_V\right) \|v_1 \wedge v_2 \wedge \cdots \wedge v_m\|_{\bigwedge^m(V)} = \left(\prod_{i=1}^m \|w_i\|_V\right)
\end{aligned}$$

The last inequality follows from Corollary 1.1.12 since $\{v_1, v_2, \ldots, v_m\}$ is an orthonormal set of vectors. Assume equality holds in (1.1.4). We will show that $w_i = \widetilde{w}_i$ with $\widetilde{w}_i$ defined as above; for then $w_1, w_2, \ldots, w_m$ are pairwise orthogonal by the Gram–Schmidt process. By definition of $\widetilde{w}_i$,

$$w_i = \widetilde{w}_i + \sum_{j=1}^{i-1} \langle w_i, v_j\rangle_V v_j.$$

Equality in (1.1.4) implies equality in (1.1.5), so that $\sum_{j=1}^{i-1} \langle w_i, v_j\rangle_V^2 = 0$ for all $i = 1, 2, \ldots, m$. Therefore, for each $i = 1, 2, \ldots, m$ we know that $\langle w_i, v_j\rangle_V = 0$ for $j = 1, 2, \ldots, i-1$. Hence, $w_i = \widetilde{w}_i$.

Conversely, if $w_1, w_2, \ldots, w_m$ are pairwise orthogonal then by definition of the norm on the exterior power

$$\begin{aligned}
\|w_1 \wedge w_2 \wedge \cdots \wedge w_m\|^2_{\bigwedge^m(V)} &= \det\left(\langle w_i, w_j\rangle_V\right)_{i,j=1\ldots m} = \det\left(\langle w_i, w_i\rangle_V \delta_{ij}\right)_{i,j=1\ldots m} \\
&= \prod_{i=1}^m \|w_i\|_V^2 = \left(\prod_{i=1}^m \|w_i\|_V\right)^2 .
\end{aligned}$$

□

Recall that the dimension of $\bigwedge^m(V)$ is the binomial coefficient $\binom{n}{m}$. Therefore, the top exterior power $\bigwedge^n(V)$ is one-dimensional, meaning it is just a real line. A non-zero n-form $\omega \in \bigwedge^n(V^*) \cong (\bigwedge^n(V))^*$ is called a ***volume (or orientation) form*** for V. In the two-dimensional case we will call an element of $\bigwedge^2(V^*)$ an ***area form*** for V. A volume form provides the one-dimensional real line $\bigwedge^n(V)$ with an orientation in a natural way by proposing that $v_1 \wedge v_2 \wedge \cdots \wedge v_n$ is ***positively oriented*** if

$$\omega(v_1 \wedge v_2 \wedge \cdots \wedge v_n) > 0.$$

Two volume forms ω and ω' establish the same orientation if and only if the factor by which they differ is positive. In particular, given a basis $\{e_1, e_2, \ldots, e_n\}$ for V, denote its dual basis $\{\varepsilon^1, \varepsilon^2, \ldots, \varepsilon^n\}$. Then the n-form $\omega_e = \varepsilon^1 \wedge \varepsilon^2 \wedge \cdots \wedge \varepsilon^n$ is a volume form such that $\omega_e(e_1 \wedge e_2 \wedge \cdots \wedge e_n) = 1$. It is called the ***standard volume form***. Since $\bigwedge^n(V^*)$ is one-dimensional, every other volume form ω is a real multiple of the standard volume form ω_e. Conversely, given an arbitrary volume form ω one can choose a basis $\{e_i'\}_{i=1}^m$ for V such that $\omega(e_1' \wedge e_2' \wedge \cdots \wedge e_n') = 1$.

Let $\{e_1, e_2, \ldots, e_n\} \subset V$ and $\{v_1, v_2, \ldots, v_n\} \subset V$ be two bases of V. Then the n-vectors $v_1 \wedge v_2 \wedge \cdots \wedge v_n$ and $e_1 \wedge e_2 \wedge \cdots \wedge e_n$ are ***consistently oriented*** with respect to a volume form $\omega \in \bigwedge^n(V^*)$ if

$$\operatorname{sgn}(\omega(v_1 \wedge v_2 \wedge \cdots \wedge v_n)) = \operatorname{sgn}(\omega(e_1 \wedge e_2 \wedge \cdots \wedge e_n)).$$

Note that this can always be achieved by switching v_1 and v_2 if necessary. Lemma 1.1.7 (v) and the properties of the norm on $\bigwedge^n(V)$ imply

$$\frac{\|v_1 \wedge v_2 \wedge \cdots \wedge v_n\|_{\bigwedge^n(V)}}{\|e_1 \wedge e_2 \wedge \cdots \wedge e_n\|_{\bigwedge^n(V)}} = |\det(T)| = \frac{|\omega(v_1 \wedge v_2 \wedge \cdots \wedge v_n)|}{|\omega(e_1 \wedge e_2 \wedge \cdots \wedge e_n)|}$$

wherein $T \colon V \to V$ is the linear map that sends e_i to v_i. If $e_1 \wedge e_2 \wedge \ldots \wedge e_n$ is a positively oriented unit n-vector then

$$\|v_1 \wedge v_2 \wedge \cdots \wedge v_n\|_{\bigwedge^n(V)} = \frac{|\omega(v_1 \wedge v_2 \wedge \cdots \wedge v_n)|}{\omega(e_1 \wedge e_2 \wedge \cdots \wedge e_n)} \tag{1.1.6}$$

and if, in addition, $\omega = \omega_e$ is the standard volume form corresponding to $\{e_1, e_2, \ldots, e_n\}$ then

$$\|v_1 \wedge v_2 \wedge \cdots \wedge v_n\|_{\bigwedge^n(V)} = |\omega(v_1 \wedge v_2 \wedge \cdots \wedge v_n)|. \tag{1.1.7}$$

In Euclidean space the following result gives meaning to the name "volume form".

Proposition 1.1.15 ([For09, (5.3) Beispiel, p. 48])
Let $v_1, v_2, \ldots, v_n \in \mathbb{R}^n$ be linearly independent vectors and P the parallelotope spanned by these vectors, that is, $P := \left\{ \sum_{i=1}^n t_i v_i \;\middle|\; t_i \in [0,1] \right\}$. Then $\mathcal{L}^n(P) = \|v_1 \wedge v_2 \wedge \cdots \wedge v_n\|_{\bigwedge^n(\mathbb{R}^n)}$ where $\mathcal{L}^n$ is the n-dimensional Lebesgue measure on $\mathbb{R}^n$.

Proof: The standard basis $\{e_1, e_2, \ldots, e_n\} \subset \mathbb{R}^n$ is orthonormal with respect to the standard Euclidean scalar product. Hence, $\|e_1 \wedge e_2 \wedge \cdots \wedge e_n\|_{\bigwedge^n(\mathbb{R}^n)} = 1$ by Corollary 1.1.12. Consider the linear map $T \colon \mathbb{R}^n \to \mathbb{R}^n$ that sends e_i to v_i. Then P is the image of the unit cube $[0,1]^n \subset \mathbb{R}^n$ under this mapping. Recall that the n-dimensional Lebesgue measure of the unit cube is 1. Further, the differential of a linear map is a linear map which has the same matrix representation. Now observe that by the change of variables theorem (see e.g. [For09, §13, Satz 2, p. 120]) the n-dimensional

Lebesgue measure of P equals

$$\begin{aligned}\mathcal{L}^n(P) &= \int_{T([0,1]^n)} 1 \, \mathrm{d}\mathcal{L}^n = \int_{[0,1]^n} |\det dT| \, \mathrm{d}\mathcal{L}^n = |\det T| \mathcal{L}^n([0,1]^n) \\ &= |\det T| = \frac{\|v_1 \wedge v_2 \wedge \cdots \wedge v_n\|_{\bigwedge^n(\mathbb{R}^n)}}{\|e_1 \wedge e_2 \wedge \cdots \wedge e_n\|_{\bigwedge^n(\mathbb{R}^n)}} = \|v_1 \wedge v_2 \wedge \cdots \wedge v_n\|_{\bigwedge^n(\mathbb{R}^n)}.\end{aligned}$$

□

Hence, by using (1.1.6) and Proposition 1.1.15, we find that the volume form applied to some $v_1 \wedge v_2 \wedge \cdots \wedge v_n$ gives the (signed) n-dimensional Lebesgue measure of the parallelotope spanned by the vectors $v_1, v_2, \ldots, v_n$ with respect to some normalisation. This normalisation corresponds to the assignment of a value for the volume of the "standard parallelotope". This geometric interpretation justifies the given terminology.

The following considerations are needed in the application of multilinear algebra in Chapter 2. Given a volume form $\omega \in \bigwedge^n(V^*)$, we want to to show that there is a ***Riesz-type isomorphism*** between V and its dual space V^* by using the volume form (instead of the inner product). Less vaguely, let us define a map $\tilde{\iota}_\omega \colon \bigwedge^{n-1}(V) \to V^*$ by

$$\tilde{\iota}_\omega(\sigma)(v) := \omega(\sigma \wedge v) \tag{1.1.8}$$

where $\sigma \in \bigwedge^{n-1}(V), v \in V$.

Proposition 1.1.16

The map $\tilde{\iota}_\omega \colon \bigwedge^{n-1}(V) \to V^*$ defined by (1.1.8) is an isomorphism.

Proof: First, note that both $\bigwedge^{n-1}(V)$ and V^* are n-dimensional. Clearly $\tilde{\iota}_\omega$ is linear because for arbitrary but fixed $v \in V$ we know that

$$\tilde{\iota}_\omega(a\sigma + \sigma')(v) = \omega((a\sigma + \sigma') \wedge v) = a\omega(\sigma \wedge v) + \omega(\sigma' \wedge v) = a\tilde{\iota}_\omega(\sigma)(v) + \tilde{\iota}_\omega(\sigma')(v)$$

where we used the bilinearity of the wedge product (Lemma 1.1.7 (i)) and the linearity of a volume form. Let $\{v_1, v_2, \ldots v_n\}$ be a basis for V. Any $\sigma \in \bigwedge^{n-1}(V)$ can be written as a linear combination of basis $(n-1)$-vectors $\sigma = \sum_{j=1}^n \alpha^j v_1 \wedge v_2 \wedge \cdots \wedge \widehat{v}_j \wedge \cdots \wedge v_n$ where the hat indicates that v_j is to be omitted. Let $\sigma \in \bigwedge^{n-1}(V)$ be such that $\omega(\sigma \wedge v) = 0$ for all $v \in V$. Note that if $i \neq j$ then $v_1 \wedge v_2 \wedge \cdots \wedge \widehat{v}_j \wedge \cdots \wedge v_n \wedge v_i = 0$. Thus, we have the following n equations

$$\begin{aligned}0 = \omega(\sigma \wedge v_i) &= \sum_{j=1}^n \alpha^j \omega(v_1 \wedge v_2 \wedge \cdots \wedge \widehat{v}_j \wedge \cdots \wedge v_n \wedge v_i) \\ &= \alpha^i \omega(v_1 \wedge v_2 \wedge \cdots \wedge \widehat{v}_i \wedge \cdots \wedge v_n \wedge v_i)\end{aligned}$$

$$= (-1)^{n-i}\alpha^i\omega(v_1 \wedge v_2 \wedge \cdots \wedge v_n)$$

for $i = 1, 2, \ldots, n$. Because ω is not the zero form and $v_1 \wedge v_2 \wedge \cdots \wedge v_n$ spans $\bigwedge^n(V)$, we find that $\alpha^i = 0$ for all $i = 1, 2, \ldots, n$, that is, $\sigma = 0$. Hence, the kernel of $\tilde{\iota}_\omega$ is trivial and $\tilde{\iota}_\omega$ is injective.

Let $l \in V^*$ be arbitrary. For $j = 1, 2, \ldots, n$ define the coefficients

$$\alpha^j := (-1)^{n-j}\frac{l(v_j)}{\omega(v_1 \wedge v_2 \wedge \cdots \wedge v_n)}$$

and set $\sigma := \sum_{j=1}^n \alpha^j v_1 \wedge v_2 \wedge \cdots \wedge \widehat{v}_j \wedge \cdots \wedge v_n$. Then $\tilde{\iota}_\omega(\sigma) = l$ since for arbitrary $v = \sum_{i=1}^n \beta^i v_i \in V$ it holds that $\omega(v_1 \wedge v_2 \wedge \cdots \wedge \widehat{v}_j \wedge \cdots \wedge v_n \wedge v) = (-1)^{n-j}\beta^j\omega(v_1 \wedge v_2 \cdots \wedge v_n)$ and thus

$$\begin{aligned}
\tilde{\iota}_\omega(\sigma)(v) = \omega(\sigma \wedge v) &= \sum_{j=1}^n \alpha^j\omega(v_1 \wedge v_2 \wedge \cdots \wedge \widehat{v}_j \wedge \cdots \wedge v_n \wedge v) \\
&= \sum_{j=1}^n \frac{(-1)^{n-j}l(v_j)}{\omega(v_1 \wedge v_2 \wedge \cdots \wedge v_n)}\omega(v_1 \wedge v_2 \wedge \cdots \wedge \widehat{v}_j \wedge \cdots \wedge v_n \wedge v) \\
&= \frac{1}{\omega(v_1 \wedge v_2 \wedge \cdots \wedge v_n)}\sum_{j=1}^n l(v_j)\beta^j\omega(v_1 \wedge v_2 \wedge \cdots \wedge v_n) \\
&= \sum_{j=1}^n l(v_j)\beta^j = l\left(\sum_{j=1}^n \beta^j v_j\right) = l\left(v\right),
\end{aligned}$$

that is, $\tilde{\iota}_\omega$ is also surjective and thereby an isomorphism. □

Since $\dim\left(\bigwedge^m(V)\right) = \binom{n}{m} = \binom{n}{n-m} = \dim\left(\bigwedge^{n-m}(V)\right)$ we know these two spaces are isomorphic as vector spaces. This holds true generally for same-dimensional vector spaces, but the isomorphism is not necessarily canonical (that is, basis-independent). In this particular case however, the so-called ***Hodge dual*** $\star\colon \bigwedge^m(V) \to \bigwedge^{n-m}(V)$ indeed provides a canonical isomorphism between $\bigwedge^m(V)$ and $\bigwedge^{n-m}(V)$. Let us only state that the inner product structure and a fixed orientation is needed to show that the Hodge dual is an isomorphism. It is completely determined by the equation

$$(\star\lambda) \wedge \theta = \langle\lambda, \theta\rangle_{\bigwedge^m(V)}\varepsilon \tag{1.1.9}$$

where $\lambda, \theta \in \bigwedge^m(V)$ are arbitrary and $\varepsilon \in \bigwedge^n(V)$ with $\|\varepsilon\|_{\bigwedge^n(V)} = 1$ is the positively oriented unit n-vector (see [BG68, §2.22, pp. 108–110] or [Fla63, §2.7, p. 15]).

Using the Hodge dual for $m = 1$ together with Proposition 1.1.16 yields the isomorphism

$$\iota_\omega = \tilde{\iota}_\omega \circ \star\colon V \to V^*. \tag{1.1.10}$$

Note that we only needed an inner product for the Hodge dual – Proposition 1.1.16 holds more generally in finite-dimensional normed vector spaces V. In particular, if the dimension $\dim(V) = n = 2$ then the isomorphism $\tilde{\iota}_\omega$ suffices for our needs and no inner product on V is required.

Let us define an n-vector $\omega^* := (\iota_\omega)^* \omega \in \bigwedge^n(V) \cong \bigwedge^n((V^*)^*)$ as the pullback of the volume form ω by ι_ω, that is,

$$\omega^*(\varphi^1 \wedge \varphi^2 \wedge \cdots \wedge \varphi^n) := \omega\left((\iota_\omega)^{-1}(\varphi^1) \wedge (\iota_\omega)^{-1}(\varphi^2) \wedge \cdots \wedge (\iota_\omega)^{-1}(\varphi^n)\right). \tag{1.1.11}$$

for $\varphi^1 \wedge \varphi^2 \wedge \cdots \wedge \varphi^n \in GC_n(V^*)$ and extend linearly to $\bigwedge^n(V^*)$. This is well-defined because ω itself is an n-form and thus alternating. In addition, ω^* is non-zero because ι_ω is an isomorphism. Therefore, ω^* is a volume form on the dual space V^*. It will be called the ***dual volume form*** with respect to ω.

Lemma 1.1.17

Let $\{e_1, e_2, \ldots, e_n\}$ be an orthonormal basis for V and $\omega = \omega_e$ the standard volume form. Denote the dual basis on V^* by $\{\varepsilon^1, \varepsilon^2, \ldots, \varepsilon^n\}$. Then $\iota_\omega(e_i) = \varepsilon^i$ for all $i = 1, 2, \ldots, n$ and

$$\omega^*(\varepsilon^1 \wedge \varepsilon^2 \wedge \cdots \wedge \varepsilon^n) = 1$$
$$\left\|\varepsilon^1 \wedge \varepsilon^2 \wedge \cdots \wedge \varepsilon^n\right\|_{\bigwedge^n(V^*)} = 1.$$

Proof: The element $e_1 \wedge e_2 \wedge \ldots \wedge e_n$ is the positively oriented unit n-vector of $\bigwedge^n(V)$ since its components form an orthonormal basis of V. The first assertion then follows from (1.1.9) since

$$\begin{aligned}
\iota_\omega(e_i)(e_j) = (\tilde{\iota}_\omega \circ \star)(e_i)(e_j) &= \tilde{\iota}_\omega(\star e_i)(e_j) \\
&= \omega(\star e_i \wedge e_j) \\
&= \omega\left(\langle e_i, e_j\rangle_V e_1 \wedge e_2 \wedge \cdots \wedge e_n\right) \\
&= \delta^i_j \omega\left(e_1 \wedge e_2 \wedge \cdots \wedge e_n\right) = \delta^i_j
\end{aligned}$$

for all $i, j = 1, 2, \ldots, n$. Then

$$\begin{aligned}
\omega^*(\varepsilon^1 \wedge \varepsilon^2 \wedge \cdots \wedge \varepsilon^n) &= \omega\left((\iota_\omega)^{-1}(\varepsilon^1) \wedge (\iota_\omega)^{-1}(\varepsilon^2) \wedge \cdots \wedge (\iota_\omega)^{-1}(\varepsilon^n)\right) \\
&= \omega\left(e_1 \wedge e_2 \wedge \cdots \wedge e_n\right) = 1.
\end{aligned}$$

If $\{e_1, e_2, \ldots, e_n\}$ is an orthonormal basis for V then the dual basis $\{\varepsilon^1, \varepsilon^2, \ldots, \varepsilon^n\}$ is orthonormal in V^* by virtue of the Riesz representation theorem. The definition of the norm on $\bigwedge^n(V^*)$ implies the final statement. $\square$

Proposition 1.1.18 (Isometric property of $\boldsymbol{\iota_\omega}$)
Let $\{e_1, e_2, \ldots, e_n\}$ be an orthonormal basis for V and $\omega = \omega_e$ the standard volume form. The isomorphism ι_ω given by (1.1.10) is isometric in the following sense:

$$\|\iota_\omega(v_1) \wedge \iota_\omega(v_2) \wedge \cdots \wedge \iota_\omega(v_n)\|_{\bigwedge^n(V^*)} = \|v_1 \wedge v_2 \wedge \cdots \wedge v_n\|_{\bigwedge^n(V)}$$

for all $v_1 \wedge v_2 \wedge \cdots \wedge v_n \in \bigwedge^n(V)$.

Proof: Recall that $v_1 \wedge v_2 \wedge \cdots \wedge v_n \neq 0 \in \bigwedge^n(V)$ if and only if $v_1, v_2, \ldots, v_n \in V$ are linearly independent. So, because ι_ω is an isomorphism, the set $\{\iota_\omega(v_1), \iota_\omega(v_2), \ldots, \iota_\omega(v_n)\}$ is a basis for V^* if $v_1 \wedge v_2 \wedge \cdots \wedge v_n \neq 0 \in \bigwedge^n(V)$. Let $T\colon V^* \to V^*$ be the map that sends ε^i to $\iota_\omega(v_i)$ where ε^i is the dual basis vector to e_i. Lemma 1.1.7 (v) and Lemma 1.1.17 yield

$$\begin{aligned}
\|\iota_\omega(v_1) \wedge \iota_\omega(v_2) \wedge \cdots \wedge \iota_\omega(v_n)\|_{\bigwedge^n(V^*)} &= \frac{\|\iota_\omega(v_1) \wedge \iota_\omega(v_2) \wedge \cdots \wedge \iota_\omega(v_n)\|_{\bigwedge^n(V^*)}}{\|\varepsilon^1 \wedge \varepsilon^2 \wedge \cdots \wedge \varepsilon^n\|_{\bigwedge^n(V^*)}} \\
&= |\det T| \\
&= \left| \frac{\omega^*\left(\iota_\omega(v_1) \wedge \iota_\omega(v_2) \wedge \cdots \wedge \iota_\omega(v_n)\right)}{\omega^*\left(\varepsilon^1 \wedge \varepsilon^2 \wedge \cdots \wedge \varepsilon^n\right)} \right| \\
&= \left|\omega^*\left(\iota_\omega(v_1) \wedge \iota_\omega(v_2) \wedge \cdots \wedge \iota_\omega(v_n)\right)\right| \\
&= |\omega(v_1 \wedge v_2 \wedge \cdots \wedge v_n)| \\
&= \|v_1 \wedge v_2 \wedge \cdots \wedge v_n\|_{\bigwedge^n(V)}.
\end{aligned}$$

The last two equalities follow from the definition of ω^* and (1.1.7). $\square$

We state a technical result on the norms of n-vectors which is needed later. For the next lemma, suppose V is an n-dimensional inner product space. Further, let $\omega \in \bigwedge^n(V^*)$ be a volume form and $e \in \bigwedge^n(V)$ the unit n-vector which is positively oriented with respect to the orientation induced by ω (that is, $\|e\|_{\bigwedge^n(V)} = 1$ and $\omega(e) > 0$). Notice that the index set $\mathcal{I}$ in the following lemma is slightly different to our usage of it so far.

Lemma 1.1.19
Let $\{v_1, v_2, \ldots, v_N\} \subset V$ be a collection of vectors for arbitrary $N \in \mathbb{N}$. Suppose that for any strictly increasing multi-index $I \in \mathcal{I} := \{(i_1, i_2, \ldots, i_n) \mid 1 \leq i_1 < i_2 < \ldots < i_n \leq N\}$ of length n the n-vectors $v_{i_1} \wedge v_{i_2} \wedge \cdots \wedge v_{i_n}$ are pairwise consistently oriented. Let $m \in \{1, 2, \ldots, n\}$ and fix a strictly increasing

multi-index $J = (j_{m+1}, j_{m+1}, \ldots, j_n)$ *of length* $n-m$ *such that* $m \le j_{m+1}$ *and* $j_n \le N$*. Then*

$$\left\| \sum_{\substack{I\in\mathcal{I}\\ i_m<j_{m+1}}} v_{i_1}\wedge\cdots\wedge v_{i_m}\wedge v_{j_{m+1}}\wedge\cdots\wedge v_{j_n} \right\|_{\bigwedge^n(V)} = \sum_{\substack{I\in\mathcal{I}\\ i_m<j_{m+1}}} \left\| v_{i_1}\wedge\cdots\wedge v_{i_m}\wedge v_{j_{m+1}}\wedge\cdots\wedge v_{j_n} \right\|_{\bigwedge^n(V)}.$$

In particular,

$$\left\| \sum_{I\in\mathcal{I}} v_{i_1}\wedge v_{i_2}\wedge\cdots\wedge v_{i_n} \right\|_{\bigwedge^n(V)} = \sum_{I\in\mathcal{I}} \| v_{i_1}\wedge v_{i_2}\wedge\cdots\wedge v_{i_n} \|_{\bigwedge^n(V)}. \tag{1.1.12}$$

Proof: The second assertion follows from the first for $m = n$. If $N < n$ then both sides of the equation are zero (because $\mathcal{I} = \varnothing$). Therefore, suppose $N \ge n$. Since $\bigwedge^n(V)$ is one-dimensional we can write $v_{i_1}\wedge v_{i_2}\wedge\cdots\wedge v_{i_n} = c_{i_1 i_2\ldots i_n} e$ for some coefficents $c_{i_1 i_2\ldots i_n} \in \mathbb{R}$. All of the coefficients $c_{i_1 i_2\ldots i_n} = \omega(v_{i_1}\wedge v_{i_2}\wedge\cdots\wedge v_{i_n})/\omega(e)$ are either positive or negative for all $I \in \mathcal{I}$ because the n-vectors $v_{i_1}\wedge v_{i_2}\wedge\cdots\wedge v_{i_n}$ are consistently oriented. In other words, $\operatorname{sgn}(c_{i_1 i_2\ldots i_n})$ is constant for any possible multi-index. Then

$$\begin{aligned}
&\sum_{\substack{I\in\mathcal{I}\\ i_m<j_{m+1}}} \| v_{i_1}\wedge\cdots\wedge v_{i_m}\wedge v_{j_{m+1}}\wedge\cdots\wedge v_{j_n} \|_{\bigwedge^n(V)} \\
&\qquad = \sum_{\substack{I\in\mathcal{I}\\ i_m<j_{m+1}}} \left\| c_{i_1\ldots i_m, j_{m+1}\ldots j_n} e \right\|_{\bigwedge^n(V)} = \sum_{\substack{I\in\mathcal{I}\\ i_m<j_{m+1}}} |c_{i_1\ldots i_m, j_{m+1}\ldots j_n}| \|e\|_{\bigwedge^n(V)} \\
&\qquad = \sum_{\substack{I\in\mathcal{I}\\ i_m<j_{m+1}}} |c_{i_1\ldots i_m, j_{m+1}\ldots j_n}| = \left| \sum_{\substack{I\in\mathcal{I}\\ i_m<j_{m+1}}} |c_{i_1\ldots i_m, j_{m+1}\ldots j_n}| \right| \\
&\qquad = \left| \sum_{\substack{I\in\mathcal{I}\\ i_m<j_{m+1}}} \underbrace{\operatorname{sgn}\left(c_{i_1\ldots i_m, j_{m+1}\ldots j_n}\right)}_{=\pm1,\ \text{const}} |c_{i_1\ldots i_m, j_{m+1}\ldots j_n}| \right| = \left| \sum_{\substack{I\in\mathcal{I}\\ i_m<j_{m+1}}} c_{i_1\ldots i_m, j_{m+1}\ldots j_n} \right|
\end{aligned}$$

and on the other hand

$$\begin{aligned}
&\left\| \sum_{\substack{I\in\mathcal{I}\\ i_m<j_{m+1}}} v_{i_1}\wedge\cdots\wedge v_{i_m}\wedge v_{j_{m+1}}\wedge\cdots\wedge v_{j_n} \right\|_{\bigwedge^n(V)} \\
&\qquad = \left\| \sum_{\substack{I\in\mathcal{I}\\ i_m<j_{m+1}}} c_{i_1\ldots i_m, j_{m+1}\ldots j_n} e \right\|_{\bigwedge^n(V)} = \left| \sum_{\substack{I\in\mathcal{I}\\ i_m<j_{m+1}}} c_{i_1\ldots i_m, j_{m+1}\ldots j_n} \right| \|e\|_{\bigwedge^n(V)}
\end{aligned}$$

$$= \left| \sum_{\substack{I \in \mathcal{I} \\ i_m < j_{m+1}}} c_{i_1 \dots i_m, j_{m+1} \dots j_n} \right|. \qquad \square$$

This section concludes with the special case $n = 2$ for the previous result.

Corollary 1.1.20

If $\dim(V) = n = 2$ *in the setting of Lemma 1.1.19 then the formulae simplify to*

$$\left\| \sum_{i=1}^{j-1} v_i \wedge v_j \right\|_{\bigwedge^2(V)} = \sum_{i=1}^{j-1} \|v_i \wedge v_j\|_{\bigwedge^2(V)}$$

for $j = 2, 3, \dots, N$ *and*

$$\left\| \sum_{1 \le i < j \le N} v_i \wedge v_j \right\|_{\bigwedge^2(V)} = \sum_{1 \le i < j \le N} \|v_i \wedge v_j\|_{\bigwedge^2(V)}.$$

1.2 Convex geometry

The scope of this section is to introduce basic concepts of convex geometry. In particular, we define polytopes – a special type of convex sets. Further, we describe the support function of a convex set and state some results for the special case of polytopes. Subsequently, the duality between the vertex and the half-space representation of polytopes is outlined. Further, we prove the Minkowski inequality for convex bodies and use a geometric argument to establish a well-known explicit formula for a specific mixed volume. Finally, we conclude this section with a result on the maxima of convex functions on polytopes. The results in this section are borrowed from the textbooks of Bonnesen and Fenchel [BF71], Brøndsted [Brø83], Grünbaum [Grü03], Matoušek [Mat02], Rockafellar [Roc70] and Thompson [Tho96, Chapter 0, Chapter 2]. Throughout the section $(V, \|\cdot\|_V)$ will denote an n-dimensional normed vector space.

1.2.1 Convex sets and polytopes

A non-empty subset C of the vector space V is said to be ***convex*** if $\lambda x + (1-\lambda)y \in C$ whenever $x, y \in C$ and $\lambda \in [0,1]$.

The ***segment joining x and y*** is defined as the set $\{\lambda x + (1-\lambda)y \mid \lambda \in [0,1]\}$ and denoted as $[x, y]$. Thus, as a more geometrical characterisation of the previous definition, a set C is convex if and only if it contains all segments joining any two of its points.

Suppose A and B are convex sets. The ***Minkowski sum*** of A and B is

$$A + B := \{\, a + b \mid a \in A, b \in B \,\}.$$

For any non-negative real number α, define further

$$\alpha A := \{\, \alpha a \mid a \in A \,\}.$$

If $A = \{a\}$ is a singleton then we usually write $a + B$ instead of $\{a\} + B$. Note that the decomposition of a vector $v \in A + B$ as a sum of vectors in A and B is, in general, not unique.

Lemma 1.2.1
Suppose A and B are convex sets and $\alpha \geq 0$ is a real number. Then $\alpha A + B$ is a convex set.

Proof: For $i = 1, 2$ let $x_i = \alpha a_i + b_i$ where $a_i \in A$ and $b_i \in B$ and $t \in [0,1]$. Then

$$tx_1 + (1-t)x_2 = \alpha(ta_1 + (1-t)a_2) + (tb_1 + (1-t)b_2)$$

which is an element of $\alpha A + B$ because A and B are convex. $\square$

Using induction we can easily generalise the previous result and show that arbitrary finite linear combinations of convex sets are convex. Furthermore, the associativity and commutativity of the vector operations in V translate to the respective properties for linear combinations of convex sets. We can also show the following distributive law.

Lemma 1.2.2

Let $\alpha, \beta_1, \beta_2 \geq 0$ be non-negative real numbers and suppose A_1, A_2 and $B \subset V$ are convex sets. Then

$$\alpha(A_1 + A_2) = \alpha A_1 + \alpha A_2$$
$$(\beta_1 + \beta_2)B = \beta_1 B + \beta_2 B.$$

Proof: The first assertion follows from the distributive law on V since $x \in \alpha(A_1 + A_2)$ if and only if there are $a_i \in A_i$ such that $x = \alpha(a_1 + a_2) = \alpha a_1 + \alpha a_2$ which is an element of $\alpha A_1 + \alpha A_2$. Similarly, if $y \in (\beta_1 + \beta_2)B$ then $y = (\beta_1 + \beta_2)b = \beta_1 b + \beta_2 b$ and thus, $y \in \beta_1 B + \beta_2 B$. Conversely, let $y = \beta_1 b_1 + \beta_2 b_2 \in \beta_1 B + \beta_2 B$. Because β_1 and β_2 are non-negative, the numbers $\frac{\beta_1}{\beta_1+\beta_2}$ and $\frac{\beta_2}{\beta_1+\beta_2}$ are non-negative and add to 1. Then the convexity of B implies $\frac{\beta_1}{\beta_1+\beta_2}b_1 + \frac{\beta_2}{\beta_1+\beta_2}b_2 \in B$ and thus

$$y = \beta_1 b_1 + \beta_2 b_2 = (\beta_1 + \beta_2)\left(\frac{\beta_1}{\beta_1 + \beta_2}b_1 + \frac{\beta_2}{\beta_1 + \beta_2}b_2\right) \in (\beta_1 + \beta_2)B. \qquad \square$$

The reason we defined αA only for non-negative scalars is precisely because of this distributive law which only applies to non-negative scalars. Also, for many of the sets that we will consider in Chapter 2, multiplication by a negative scalar will be irrelevant because they will be symmetric in the following sense.

A set A is ***symmetric with respect to the origin* $\mathbf{0} \in V$** if $(-1)A = \{-a \mid a \in A\} = A$. Henceforth, if we use the adjective "symmetric" it always means symmetric with respect to the origin.

We can see as follows that any intersection of convex sets is convex. If x and y are elements of all the members of the intersection, so is the segment joining x and y because each of the sets is convex. But then the line segment already is contained in the intersection itself. Therefore it is fitting to consider the intersection of the family of convex sets containing a given (not necessarily convex) set A. By construction, this is the smallest convex set containing A. It is called the ***convex hull*** of A and denoted by $\operatorname{conv}(A)$. Due to Carathéodory [DGK63, pp.103+115] the convex hull of a set $A \subset V$ is characterised by

$$\operatorname{conv}(A) = \left\{ \sum_{i=1}^{n+1} \lambda_i a_i \,\middle|\, a_i \in A, \lambda_i \geq 0, \sum_{i=1}^{n+1} \lambda_i = 1 \right\}. \tag{1.2.13}$$

One can show that the convex hull operation preserves inclusions, in other words $\operatorname{conv}(X) \subset \operatorname{conv}(Y)$ if $X \subset Y$.

A linear subspace $W \subset V$ is trivially convex. Clearly, a singleton $\{x\}$ is convex as well. Thus, any translate of a linear subspace, that is, $x + W$ is convex. Such a set is an ***affine subspace*** of V. Its dimension is defined as the dimension of the subspace of which it is a translate. The ***dimension of a convex set*** is the dimension of the affine subspace of smallest dimension containing the set. For example, the dimension of a cube is 3, the dimension of a triangle 2 and the dimension of a point is 0. Since any convex set is contained in the linear space V, the dimension of a convex set is bounded above by the dimension n of the ambient space V.

A convex subset E of a convex set A is said to be an ***extreme subset*** of A if, whenever $x \in E$ and $x = \lambda a_1 + (1-\lambda)a_2$ for $a_1, a_2 \in A$ and $\lambda \in (0,1)$, then $a_1, a_2 \in E$ already.

A special type of convex sets, the "polytopes", will occupy much of our focus in this section and the application later. To avoid unnecessary clutter in explicit calculations and formulae, we make the following construction. Suppose $\{a_1, a_2, \ldots, a_N\}$ and $\{b_1, b_2, \ldots, b_M\}$ are two non-empty sets of finitely many points in V. Any such two sets are in ***polytopial relation*** to one another if and only if $\operatorname{conv}(\{a_1, a_2, \ldots, a_N\}) = \operatorname{conv}(\{b_1, b_2, \ldots, b_M\})$. Clearly, this defines an equivalence relation because equality of sets is one such correspondence. The equivalence class $\big[\{a_1, a_2, \ldots, a_N\}\big]_{poly}$ can be well-ordered when ordering the number of points of each of its elements with the usual order $\leq$ of the natural numbers $\mathbb{N}$. For each equivalence class $\big[\{a_1, a_2, \ldots, a_N\}\big]_{poly}$ the well-ordering principle then yields the existence of an element with a minimal number of points.

A ***polytope*** is the convex hull of the minimal element in the equivalence class of a non-empty set of finitely many points in V, that is,

$$A_N := [a_1 a_2 \ldots a_N] := \operatorname{conv}(\{a_1, a_2, \ldots, a_N\}),$$

where N is minimal as described above. The points $a_1, a_2, \ldots, a_N$ are the ***vertices*** of A. In the special case of dimension $n = 2$, a polytope is called a ***polygon***. If there is no need for the number of indices the index N will be omitted.

If not explicitly stated otherwise a polytope in V is considered as full dimensional, that is, its dimension is n. This can always be achieved by restricting the point of view to the affine subspace containing the polytope. In doing so, the number of vertices N is bounded below by $n + 1$. This can be seen through (1.2.13).

The extreme subsets of a polytope are called the ***faces*** or ***j-faces*** if their dimension is j. The $(n-1)$-dimensional faces of a polytope are called ***facets***. Note that the 0-dimensional faces of a polytope are its vertices. One can show that the faces of a polytope are polytopes themselves ([Brø83, Theorem 7.3, p. 45]).

Lemma 1.2.3

A polytope is a convex, closed and bounded (and thus compact) set.

Proof: Let $A = [a_1 a_2 \dots a_N]$ be a polytope. By definition, a polytope is convex as the convex hull of its vertices. Note that a polytope A is contained in the ball of radius $\max_{i=1,2,\dots,N} \|a_i\|_V$ and thus bounded. Let $(x_k)_{k\in\mathbb{N}} \subset A$ be a sequence which converges to an arbitrary but fixed $x \in V$ in the norm topology of V. We need to show that $x \in A$. Any of the points x_k can be expressed as a convex combination of the vertices, that is, $x_k = \sum_{i=1}^N \lambda_i^{(k)} a_i$ where $\sum_{i=1}^N \lambda_i^{(k)} = 1$ and $\lambda_i^{(k)} \in [0,1]$. Because $[0,1]$ is a compact set there is a subsequence, say $(k_{i_1})_{i_1\in\mathbb{N}} \subset (k)_{k\in\mathbb{N}}$, such that $\left(\lambda_1^{(k_{i_1})}\right)_{i_1\in\mathbb{N}}$ converges to some $\lambda_1 \in [0,1]$. But then there is another subsequence of this subsequence, say $(k_{i_2})_{i_2\in\mathbb{N}} \subset (k_{i_1})_{i_1\in\mathbb{N}} \subset (k)_{k\in\mathbb{N}}$, such that both $\left(\lambda_1^{(k_{i_2})}\right)_{i_1\in\mathbb{N}}$ and $\left(\lambda_2^{(k_{i_2})}\right)_{i_1\in\mathbb{N}}$ converge to some λ_1 and $\lambda_2 \in [0,1]$ respectively. Iterating this process N times and renaming the final subsequence yields the convergence of $\left(\lambda_i^{(k)}\right)_{k\in\mathbb{N}}$ to $\lambda_i \in [0,1]$ for $i = 1,2,\dots,N$. Further, $\sum_{i=1}^N \lambda_i = \sum_{i=1}^N \lim_{k\to\infty} \lambda_i^{(k)} = \lim_{k\to\infty} \sum_{i=1}^N \lambda_i^{(k)} = 1$. Define $y := \sum_{i=1}^N \lambda_i a_i \in A$. Then the triangle inequality for the norm on V implies

$$\|y - x_k\|_V \leq \sum_{i=1}^N \left|\lambda_i^{(k)} - \lambda_i\right| \|a_i\|_V \xrightarrow[k\to\infty]{} 0$$

and

$$\|y - x\|_V \leq \|y - x_k\|_V + \|x_k - x\|_V \xrightarrow[k\to\infty]{} 0.$$

Thus, $x = \lim_{k\to\infty} x_k = y \in A$. □

1.2.2 Support functions

Consider an $(n-1)$-dimensional subspace W of V with basis $\{w_1, w_2, \dots, w_{n-1}\}$. Extend this to a basis $\{w_1, w_2, \dots, w_{n-1}, v\}$ of the entire space V and define a linear form $f\colon V \to \mathbb{R}$ on this basis by $f(w_i) = 0$ for $i = 1,2,\dots,n-1$ and $f(v) = 1$. Then $\ker(f) = W$. Clearly, this particular linear form f is not unique – a different choice of basis yields a different linear form. Conversely, if f is a non-zero element of the dual space V^* then its image is all of $\mathbb{R}$, because for a vector $v \notin \ker(f)$ it holds that $f(\alpha v) = \alpha f(v)$ for all $\alpha \in \mathbb{R}$. The rank-nullity-theorem then states that the kernel of f is of dimension $\dim(\ker(f)) = \dim(V) - \dim(\mathrm{im}(f)) = n - 1$. Therefore, it is appropriate to introduce the following notion.

A translate of the kernel of a non-zero linear form $f \in V^*$ is called a ***hyperplane*** and the linear form that generates it is a ***normal*** to this hyperplane. If $\|f\|_{V^*} = 1$ then f is called a ***unit normal***. Note that the normal to a hyperplane is, in general, not unique.

Remark:
Suppose V is equipped with an inner product. Then the notion of normality introduced above coincides with the usual geometrical understanding of normality. Consider a non-zero linear form $f \in V^*$. Through the Riesz representation theorem f corresponds to a vector $\bar{n} \in V$ by $f(w) = \langle w, \bar{n}\rangle_V$ for all $w \in V$. A vector is normal to a translate $x + W$ of a subspace in the usual sense if it is orthogonal to all vectors in the subspace W. But then

$$W = \{w \in V \mid \langle w, \bar{n}\rangle_V = 0\} = \{w \in V \mid f(w) = 0\} = \ker(f),$$

that is, $W + x$ is in fact a hyperplane as defined above. In this section however, we will not assume that V is equipped with an inner product because the results hold in all finite-dimensional normed spaces – which is the reason why we introduced the notion of normality for linear forms and not for vectors.

A hyperplane H can be described as a level set of its normal f because

$$\begin{aligned} H = x + \ker(f) &= \{x + v' \mid v' \in \ker(f)\} \\ &= \{v = x + v' \mid v' \in \ker(f), f(v) = f(v') + f(x) = f(x)\} \\ &= \{v \in V \mid f(v) = \alpha\} =: H_\alpha(f) \end{aligned}$$

where $\alpha := f(x)$. Given a non-zero linear form f we denote the family of hyperplanes that f is normal to by $H_\alpha(f)$ where $\alpha \in \mathbb{R}$. Note $\ker(f) = H_0(f)$. Each hyperplane $H_\alpha(f)$ bounds the two ***closed half-spaces*** $H_\alpha^+(f) := \{v \in V \mid f(v) \geq \alpha\}$ and $H_\alpha^-(f) := \{v \in V \mid f(v) \leq \alpha\}$. Both of these half-spaces are convex because f is linear and therefore any intersection of closed half-spaces is a closed convex set. Using the Hahn-Banach theorem and its geometric variants, one can prove the following partial converse.

Proposition 1.2.4 ([Tho96, Theorem 2.1.8, p. 48])
Every closed convex set is the intersection of those closed half-spaces which contain it.

A hyperplane $H := H_\alpha(f)$ is said to be a ***supporting hyperplane*** for a closed convex set K, if $K \cap H \neq \varnothing$ and either $K \subset H_\alpha^+(f)$ or $K \subset H_\alpha^-(f)$. In this case we say that the linear form f ***supports*** the convex set K at each point in $K \cap H$. Note that $H_{-\alpha}^+(-f) = H_\alpha^-(f)$ and thus, we may restrict our point of view to hyperplanes of the form $H_\alpha^-(f)$. If $H_\alpha^-(f)$ is a supporting half-space for K then $\frac{1}{\|f\|_{V^*}} f$ is called an ***outer unit normal***.

With all this notation at hand, we can introduce an important mapping corresponding to a given bounded, closed and convex set K. The function

$$h_K \colon V^* \to \mathbb{R}, f \mapsto \sup_{x \in K} f(x)$$

is called the ***support function of K***. This function has the following properties.

Lemma 1.2.5

(i) The support function is sublinear on V^*, that is,

$$h_K(\alpha f) = \alpha h_K(f), \qquad \textit{for } \alpha \geq 0$$
$$h_K(f+g) \leq h_K(f) + h_K(g).$$

(ii) The support function respects Minkowski sums, that is,

$$h_{\alpha K} = \alpha h_K, \qquad \textit{for } \alpha \geq 0$$
$$h_{K+K'} = h_K + h_{K'}.$$

(iii) If $0 \in K$ then $h_K(f) \geq 0$ and if 0 is an interior point of K then $h_K(f) > 0$ for all $f \neq 0$.

(iv) If K is symmetric then $h_K(f) = h_K(-f)$ and $h_K(f) = \sup_{x \in K} |f(x)|$.

(v) If K is the unit ball of V then $h_K = \|\cdot\|_{V^*}$.

(vi) If $K \subset K'$ then $h_K \leq h_{K'}$.

Proof: Part (i) follows from the definition of h_K. For part (ii) calculate

$$h_{\alpha K}(f) = \sup_{x \in \alpha K} f(x) = \sup_{y \in K} f(\alpha y) = \alpha \sup_{y \in K} f(y) = \alpha h_K(f)$$

and

$$\begin{aligned} h_{K+K'}(f) &= \sup_{x\in K+K'} f(x) = \sup_{y\in K, y'\in K'} f(y+y') \\ &= \sup_{y\in K, y'\in K'} f(y)+f(y') = \sup_{y\in K} f(y) + \sup_{y'\in K'} f(y') \\ &= h_K(f) + h_{K'}(f) \end{aligned}$$

for $f \in V^*$ and $\alpha \geq 0$.

To prove part (iii), suppose that $h_K(f) = \sup_{y\in K} f(y) < 0$ for some $f \in V^*$. Since $0 \in K$, we have the contradiction

$$0 = f(0) \leq \sup_{y\in K} f(y) < 0.$$

Let 0 be an interior point of K and suppose further there is $f \in V^*$ with $h_K(f) = \sup_{x\in K} f(x) = 0$, that is, $f\big|_K \equiv 0$. Since 0 is an interior point there is a radius $\varepsilon > 0$ such that the norm ball $B_\varepsilon(0)$ centered at 0 is contained in K. Moreover, $f\big|_{B_\varepsilon(0)} \equiv 0$. This already implies $f \equiv 0$ since any $0 \neq x \in V$ can be scaled by the factor $\frac{\varepsilon}{2\|x\|_V}$ such that $y := \frac{\varepsilon}{2\|x\|_V} x \in B_\varepsilon(0)$. The linearity of f then yields $f(x) = \frac{2\|x\|_V}{\varepsilon} f(y) = 0$. For part (iv), note that if K is symmetric (that is, $K = (-1)K$) then

$$\begin{aligned} h_K(-f) &= \sup_{x\in K} (-f)(x) = \sup_{x\in K} f(-x) \\ &= \sup_{-x\in K} f(x) = \sup_{x\in(-1)K} f(x) = \sup_{x\in K} f(x) \\ &= h_K(f). \end{aligned}$$

For part (v) just note that $0 \in K$. Part (iii) and the definition of the dual norm immediately yield the claim since $h_K(f) = \sup_{x\in K} |f(x)| = \|f\|_{V^*}$. Part (vi) is clear because $\sup_{x\in K} f(x) \leq \sup_{x\in K'} f(x)$ if $K \subset K'$. $\square$

Since h_K is a positively homogeneous mapping its image is completely determined by the image of linear forms of unit length. As mentioned earlier any closed convex set is contained in the intersection of its supporting half-spaces. Although very important this is merely a qualitative statement. However, the next result shows that the support function h_K completely describes these supporting half-spaces and provides a geometrical meaning to its value at a unit length linear form.

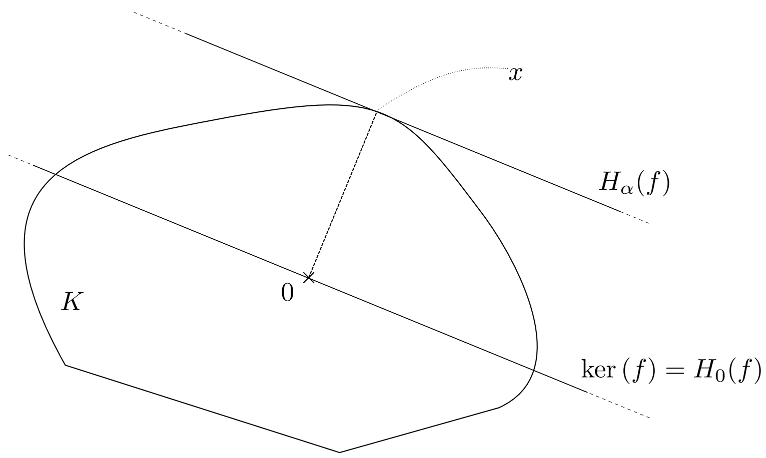


Figure 1.1: Distance of $0 \in K$ to $H_\alpha(f)$

Proposition 1.2.6

Let $f \in V^*$ be a non-zero linear form. Suppose K is a closed bounded and convex set which contains the origin as an interior point. Define $\alpha := h_K(f) > 0$. Then the hyperplane $H_\alpha(f)$ is a supporting hyperplane for K and K is contained in the half-space $H_\alpha^-(f) = \{v \in V \mid f(v) \leq h_K(f)\}$ (that is, f is an outer normal to $H_\alpha(f)$). In particular, $h_K\left(\frac{f}{\|f\|_{V^*}}\right)$ gives the distance of the origin to the hyperplane $H_\alpha(f) = \{v \in V \mid f(v) = h_K(f)\}$, that is,

$$h_K\left(\frac{f}{\|f\|_{V^*}}\right) = \operatorname{dist}(0, H_\alpha(f)). \tag{1.2.14}$$

Proof: Since K is compact as a closed and bounded set in V and f is continuous, there is $x \in K$ such that $f(x) = \sup_{y \in K} f(y) = h_K(f) = \alpha$. Then $x \in K \cap H_\alpha(f)$. Further, $K \subset H_\alpha^-(f)$ because $f(z) \leq \sup_{y \in K} f(y) = h_K(f)$ for all $z \in K$. Hence, $H_\alpha(f)$ is a supporting hyperplane for K.

Note that by the linearity of f, $\frac{1}{t} H_\alpha(f) = H_{t\alpha}(f)$ for all $t \neq 0$. Therefore, the hyperplane $H_\alpha(f)$ parametrises the open half-space $H_0^+(f)$ which is bounded by the kernel of f. Similarly, the hyperplane $H_{-\alpha}(f)$ parametrises the open half-space $H_0^-(f)$.

Then we calculate

$$\begin{aligned}
\operatorname{dist}(0, H_\alpha(f)) &= \inf_{z \in H_\alpha(f)} \|z\|_V = \alpha \inf_{z \in H_\alpha(f)} \left(\frac{\alpha}{\|z\|_V}\right)^{-1} \\
&= \alpha \left(\sup_{z \in H_\alpha(f)} \frac{\alpha}{\|z\|_V}\right)^{-1} = h_K(f) \left(\sup_{z \in H_\alpha(f)} \frac{f(z)}{\|z\|_V}\right)^{-1} \\
&= h_K(f) \left(\sup_{z \in H_\alpha(f)} f\left(\frac{z}{\|z\|_V}\right)\right)^{-1}.
\end{aligned}$$

The expression $\frac{z}{\|z\|_V}$ does not change its value if z is scaled by an arbitrary positive factor $t > 0$. Thus, we can rewrite the supremum,

$$\sup_{\substack{z\in H_\alpha(f)\\ t>0}} f\left(\frac{\frac{z}{t}}{\|\frac{z}{t}\|_V}\right) = \sup_{\substack{z\in \frac{1}{t}H_\alpha(f)\\ t>0}} f\left(\frac{z}{\|z\|_V}\right) = \sup_{\substack{z\in H_{t\alpha}(f)\\ t>0}} f\left(\frac{z}{\|z\|_V}\right) = \sup_{z\in V, f(z)>0} f\left(\frac{z}{\|z\|_V}\right).$$

Notice that

$$\sup_{z\in V, f(z)>0} f\left(\frac{z}{\|z\|_V}\right) = \sup_{z\in V, f(z)>0} -f\left(\frac{-z}{\|-z\|_V}\right) = \sup_{z\in V, f(z)<0} -f\left(\frac{z}{\|z\|_V}\right)$$

which implies

$$\sup_{z\in V, f(z)>0} f\left(\frac{z}{\|z\|_V}\right) = \sup_{z\in V, f(z)\neq 0} \left|f\left(\frac{z}{\|z\|_V}\right)\right|.$$

This finally yields

$$\begin{aligned}\operatorname{dist}(0, H_\alpha(f)) &= h_K(f)\left(\sup_{z\in V, f(z)\neq 0}\left|f\left(\frac{z}{\|z\|_V}\right)\right|\right)^{-1} = h_K(f)\left(\sup_{z\in V, z\neq 0}\left|f\left(\frac{z}{\|z\|_V}\right)\right|\right)^{-1}\\ &= h_K(f)\frac{1}{\|f\|_{V^*}} = h_K\left(\frac{f}{\|f\|_{V^*}}\right).\end{aligned}$$ □

The previous result will be used repeatedly in the proofs of explicit calculations on polytopes carried out in Section 1.2.5.

1.2.3 Polyhedra and polarity of convex sets

Recall that if $0 \in K$ is an interior point then $h_K(f) > 0$ if $f \neq 0$. Therefore,

$$H_\alpha^-(f) = \{v \in V \mid f(v) \leq h_K(f)\} = \left\{v \in V \;\middle|\; \frac{1}{h_K(f)} f(v) \leq 1\right\} = H_1^-(\tilde{f})$$

where $\tilde{f} := \frac{1}{h_K(f)} f$. Thus, we may restrict our point of view further to half-spaces described by $H_1^-(f)$. By the previous proposition, we can then describe a closed bounded and convex set which contains the origin as an interior point by

$$K = \bigcap_{f\in V^*} H_{h_K(f)}^-(f) = \bigcap_{f\in V^*} H_1^-\left(\frac{1}{h_K(f)} f\right). \tag{1.2.15}$$

A ***polyhedral set*** P is the intersection of a finite number of half-spaces, that is,

$$P = \bigcap_{i=1}^{M} H^-_{\alpha_i}(f^i)$$

for non-zero linear forms $f^i \in V^*$ and scalars $\alpha_i \in \mathbb{R}, i = 1, 2, \ldots, M$. We assume here that the number M is minimal in the sense that none of the half-spaces $H^-_{\alpha_j}(f^j)$ may be omitted in the intersection above (Brøndsted calls this an "irreducible" representation, see [Brø83, p. 52]). One of the main theorems in convexity theory of polytopes is the following statement.

Theorem 1.2.7 ([Brø83, Chapter 9, Theorem 9.2])
A non-empty subset P of V is a polytope if and only if it is a bounded polyhedral set.

This theorem gives two representations of a polytope – the vertex representation on the one hand and the half-space representation on the other. An elegant proof, using only elementary arguments, can be found in [Mat02, pp. 84–85]. There is a wide range of results on the combinatorial properties of such sets – the text books [Brø83] and [Grü03] provide good references for this topic. Regarding combinatorics of polytopes, let us only note here that a symmetric polytope has an even number of vertices and an even number of facets, which can be deduced by using the vertex and half-space representation respectively.

For the application in the next chapter we need another concept that plays an important role in convexity theory. For each closed convex set K we can define its ***polar set*** K° in the dual space V^* by

$$\begin{aligned} K^\circ &:= \{f \in V^* \mid f(x) \leq 1 \text{ for all } x \in K\} \\ &= \bigcap_{x \in K} \{f \in V^* \mid f(x) \leq 1\}. \end{aligned} \tag{1.2.16}$$

It is clear that the polar set is non-empty since $0 \in K^\circ$. The polar operation reverses set inclusions, for if $K_1 \subset K_2$ then $K_2^\circ \subset K_1^\circ$. Moreover, the next properties hold to be true.

Lemma 1.2.8
The polar set K° can be described through the support function of K as $K^\circ = \{f \in V^* \mid h_K(f) \leq 1\}$. If B is the unit ball in V then B° is the unit ball in V^*. The polar set is a convex and closed set. If K is bounded then $0 \in V^*$ is an interior point of K°. Furthermore, if $0 \in V$ is an interior point of K then K° is bounded.

Proof: Let $f \in K^\circ$, that is, $f(x) \leq 1$ for all $x \in K$. But then this also holds for the least upper bound, so $h_K(f) = \sup_{x\in K} f(x) \leq 1$. Conversely, if $f \in V^*$ such that $h_K(f) = \sup_{x\in K} f(x) \leq 1$, then $f(x) \leq 1$ for all $x \in K$, that is, $f \in K^\circ$. If B is the unit ball of V then part (v) of Lemma 1.2.5 yields

$$B^\circ = \{f \in V^* \mid h_B(f) \leq 1\} = \{f \in V^* \mid \|f\|_{V^*} \leq 1\}.$$

For any convex combination of elements of K°

$$\big(tf + (1-t)g\big)(x) = tf(x) + (1-t)g(x) \leq t + (1-t) = 1$$

for all $x \in K$, therefore K° is convex. Further, let $(f_k)_{k\in\mathbb{N}} \subset K^\circ$ and $f_k \xrightarrow[k\to\infty]{} f \in V^*$. Then for all $x \in K$

$$\begin{aligned} f(x) &= f(x) - f_k(x) + f_k(x) \\ &\leq \|f - f_k\|_{V^*} + f_k(x) \\ &\leq \|f - f_k\|_{V^*} + 1 \xrightarrow[k\to\infty]{} 1. \end{aligned}$$

Thus, $f \in K^\circ$ and K° is closed. If $B_R(0) = RB_1(0) = RB$ is the norm ball in V of radius $R > 0$ centered at the origin, note that by Lemma 1.2.5 (v)

$$h_{B_R(0)}(f) = Rh_B(f) = R\|f\|_{V^*}$$

and because B° is the unit ball in V^* it follows that

$$\begin{aligned} (B_R(0))^\circ &= \left\{f \in V^* \;\middle|\; h_{B_R(0)}(f) \leq 1\right\} \\ &= \{f \in V^* \mid \; R\|f\|_{V^*} \leq 1\} \\ &= \left\{f \in V^* \;\middle|\; \|f\|_{V^*} \leq R^{-1}\right\} \\ &= R^{-1}B^\circ =: B^*_{R^{-1}}(0). \end{aligned}$$

In other words, $(B_R(0))^\circ$ is the norm ball in the dual space V^* of radius R^{-1} centered at the origin. If K is bounded then there is a radius $R > 0$ such that $K \subset B_R(0)$. As the polar operation reverses inclusions, we have $B^*_{R^{-1}}(0) \subset K^\circ$, that is, 0 is an interior point of K°. Conversely, if $B_R(0) \subset K$ for some $R > 0$ the same argumentation yields $K^\circ \subset B^*_{R^{-1}}(0)$, that is, K° is bounded. □

Proposition 1.2.9

Let P be a polytope which contains the origin as an interior point. Then P° is a polytope.

PROOF: Suppose $P = \operatorname{conv}(a_1, a_2, \dots, a_N)$. Then P is closed bounded and convex due to Lemma 1.2.3. By Lemma 1.2.8 we know that P° is also closed, convex and contains $0 \in V^*$ as an interior point. In addition,

$$P^\circ = \bigcap_{x \in P} \{f \in V^* \mid f(x) \leq 1\} = \bigcap_{i=1}^{N} \{f \in V^* \mid f(a_i) \leq 1\}.$$

The second equality follows from the fact that $a_i \in P$, so $f(a_i) \leq 1$ for $f \in P^\circ$. Conversely, suppose $f \in V^*$ such that $f(a_i) \leq 1$ for all $i = 1, 2, \dots, N$. We then have $f(x) = f\left(\sum_{i=1}^N \lambda_i a_i\right) = \sum_{i=1}^N \lambda_i f(a_i) \leq \sum_{i=1}^N \lambda_i = 1$ for all $x \in P$, that is, $f \in P^\circ$.

For any normed space the canonical mapping $\Phi\colon V \to V^{**}$, $\Phi(v)(f) := f(v)$ is linear and preserves the norm, that is, $\|\Phi(v)\|_{V^{**}} = \|v\|_V$ (see [Wer00, Satz III.3.1, p. 105]). This implies that Φ is injective, for if $\Phi(v) = 0$ then v must be zero. Since V and V^* and $V^{**} = (V^*)^*$ are all vector spaces of the same dimension n, the map Φ is already surjective by the rank-nullity theorem because $\dim(\operatorname{im}(\Phi)) = \dim(V) - \dim(\ker(\Phi)) = n - 0 = n$. Thus,

$$\{f \in V^* \mid f(a_i) \leq 1\} = \{f \in V^* \mid \Phi(a_i)(f) \leq 1\} = H_1^-(\Phi(a_i))$$

is a half-space in V^*, whence P° is a polyhedral set as the intersection of those finitely many half-spaces. Furthermore, because P contains the origin as an interior point, the polar set P° is bounded due to Lemma 1.2.8. By Theorem 1.2.7, P° then is a polytope. $\square$

More precisely (or rather "conversely" in terms of vertex and half-space representations) the following statement holds.

Lemma 1.2.10 ([Grü03, Exercise 3.4.7, p. 49] or [Mat02, Exercise 5.1.6 (b), p. 81])
If the convex polytope P (containing the origin as an interior point) is given via its half-space representation as

$$P = \bigcap_{i=1}^{M} \{v \in V \mid f^i(v) \leq 1\}$$

where $f^i \in V^*$ for $i = 1, 2, \dots, M$ then the polar set is given via the vertex representation

$$P^\circ = \operatorname{conv}\left(f^1, f^2, \dots, f^M\right).$$

PROOF: Define $Q := \operatorname{conv}\left(f^1, f^2, \dots, f^M\right) \subset V^*$. The set $Q^\circ = \{\tilde{v} \in V^{**} \mid \tilde{v}(f) \leq 1 \text{ for all } f \in Q\} \subset V^{**}$ is its polar set. We will show $Q^\circ = P$. Referring to the previous proof, recall the canonical isomorphism $\Phi\colon V \to V^{**}$ where $\Phi(v)(f) := f(v)$. Using this identification, we consider Q° to be a subset of V and

$$Q^\circ = \{v \in V \mid f(v) \leq 1 \text{ for all } f \in Q\} = \bigcap_{f \in Q} \{v \in V \mid f(v) \leq 1\}.$$

Suppose $v \in Q^\circ$, then we have in particular $f^i \in Q$ for $i = 1, 2, \ldots, M$ so that $f^i(v) \leq 1$. Conversely, suppose $v \in V$ so that $f^i(v) \leq 1$ for all $i = 1, 2, \ldots, M$. Then for all $f \in Q$ we have $f(v) = \left(\sum_{i=1}^N \lambda_i f^i\right)(v) = \sum_{i=1}^N \lambda_i f^i(v) \leq \sum_{i=1}^N \lambda_i = 1$. Thus, the last identity can be rewritten as

$$Q^\circ = \bigcap_{i=1}^{M} \{v \in V \mid f^i(v) \leq 1\} = P.$$

By taking the polar yet again, this implies $P^\circ = Q^{\circ\circ}$. (It is understood that this equality sign refers to the identification of V^* with $(V^*)^{**}$ via the mapping $\Phi^*\colon V^* \to (V^*)^{**}, \Phi^*(v)(f) = f(v)$.) Furthermore, the polytope Q is bounded (Lemma 1.2.3) so that by Lemma 1.2.8 the set Q° contains the origin as an interior point and is closed. Then $Q^{\circ\circ} = Q$ (see [Tho96, Theorem 2.2.6, p. 50]) which proves the claim – again it is understood that the equality sign refers to the identification of V^* with V^{***} via the mapping Φ^*. □

Note that the number of facets and vertices of a polytope and its polar set interchange their roles. This is one example of duality in the theory of convex polytopes and gives rise to the so-called "face-lattice" (see [Brø83, Chapter 1, §5, p. 29f]).

1.2.4 Mixed volume and the Minkowski inequality

We introduce a quantity that relates a collection of convex sets with each other – the mixed volume. For the application in Chapter 2 we prove the Minkowski inequality for convex bodies.

A ***convex body*** is a compact, convex subset of V with non-empty interior. The collection of all convex bodies in V is denoted by $\mathcal{C}_b = \mathcal{C}_b(V)$. In Lemma 1.2.3 we have seen that any full-dimensional polytope is a convex body. We need full-dimensionality here to ascertain a non-empty interior.

To motivate the notion of mixed volume, the following observation is useful. Using the homogeneity of the Hausdorff measure on V, one sees that the measure of a scalar multiple $\alpha = \alpha_1 + \ldots + \alpha_k$ of a set K is

$$\mathcal{H}^n(\alpha_1 K + \ldots + \alpha_k K) = (\alpha_1 + \ldots + \alpha_k)^n \mathcal{H}^n(K) = \left(\sum_{i_1, \ldots, i_n = 1}^{k} \alpha_{i_1} \ldots \alpha_{i_n}\right) \mathcal{H}^n(K),$$

that is, $\mathcal{H}^n(\alpha_1 K + \ldots + \alpha_k K)$ is a homogeneous polynomial of degree n in the coefficients α_i. In greater generality, one can show that this holds for a non-negative linear combination of convex bodies. In particular, for any collection of convex bodies $\{K_1, \ldots, K_k\} \subset V$ and non-negative coefficients $\alpha_i \geq 0$ the map $(\alpha_1, \ldots, \alpha_k) \mapsto \mathcal{H}^n(\alpha_1 K_1 + \ldots + \alpha_k K_k)$ is a homogeneous polynomial of degree n (see [Tho96, Theorem 2.3.6, p. 55]). Hence, it can be expressed as

$$\mathcal{H}^n(\alpha_1 K_1 + \ldots + \alpha_k K_k) = \sum_{i_1, \ldots, i_n = 1}^{k} \mathrm{V}(K_{i_1}, \ldots, K_{i_n}) \alpha_{i_1} \ldots \alpha_{i_n} \tag{1.2.17}$$

where the coefficient term $\mathrm{V}(K_{i_1}, \ldots, K_{i_n})$ is symmetric in its arguments (because the Minkowski sum in the left hand side is commutative) and is called the ***mixed volume*** of the convex bodies $K_{i_1}, \ldots, K_{i_n}$. Note that the mixed volume is always a function of n arguments – the dimension of the ambient space V – independent of the length of the linear combination. By inspection, we can see $\mathrm{V}(K, \ldots, K) = \mathcal{H}^n(K)$. Furthermore, if any of the sets K_i is translated linearly by some $x \in V$ the invariance of Hausdorff measure (on the left hand side of (1.2.17)) under translations implies that the mixed volume (on the right hand side of (1.2.17)) is invariant under translations of any of its arguments. The symmetry of mixed volume allows to reorder and collect like terms in the above polynomial expression. There are $\binom{n}{j_1, \ldots, j_k} := \binom{n}{j_1}\binom{n-j_1}{j_2} \cdots \binom{n-j_1-\ldots-j_{k-1}}{j_k}$ terms such that j_1 of the n indices $i_1, \ldots, i_n$ are equal to 1, j_2 of the remaining $n - j_1$ indices are equal to 2, etc. The mixed volume then only depends on the new indices $j_1, \ldots, j_k$ and will be denoted by

$$\mathrm{V}(K_1[j_1], \ldots, K_k[j_k]) := \mathrm{V}(K_1, K_1, \ldots, K_1, K_2, K_2, \ldots, K_2, \ldots, K_k, K_k, \ldots, K_k)$$

where K_l appears j_l times. Thus, the polynomial expression can be rewritten as

$$\mathcal{H}^n(\alpha_1 K_1 + \ldots + \alpha_k K_k) = \sum_{j_1 + \ldots + j_k = n} \binom{n}{j_1, \ldots, j_k} \mathrm{V}(K_1[j_1], \ldots, K_k[j_k]) \alpha_1^{j_1} \ldots \alpha_k^{j_k}.$$

For the Minkowski sum $sK + tK'$ of two convex bodies K and K' this formula becomes

$$\mathcal{H}^n(sK + tK') = \sum_{j=0}^{n} \binom{n}{j} \mathrm{V}(K[n-j], K'[j]) s^{n-j} t^j. \tag{1.2.18}$$

If $s = 1$ in the last expression, we conclude the following formula.

Corollary 1.2.11
The mixed volume of $K[n-1]$ *and* $K' = K'[1]$ *can be calculated by*

$$\mathrm{V}\left(K[n-1], K'\right) = \frac{1}{n} \lim_{t \to 0} \frac{\mathcal{H}^n\left(K + tK'\right) - \mathcal{H}^n\left(K\right)}{t}. \tag{1.2.19}$$

There are several inequalities that relate the different mixed volumes of a collection of convex bodies. An important one is the Brunn-Minkowski inequality (see [Tho96, Theorem 2.3.5, p. 55]) which will be included in the proof below. We will need the following consequence of this inequality in the application in Chapter 2.

Proposition 1.2.12 (Minkowski inequality for convex bodies, [BF71, §49, p. 91])
If K *and* K' *are convex bodies then*

$$\big(\mathrm{V}\left(K[n-1], K'\right)\big)^n \geq \big(\mathcal{H}^n\left(K\right)\big)^{n-1} \mathcal{H}^n\left(K'\right).$$

Proof: The Brunn-Minkowski inequality states that

$$\big(\mathcal{H}^n\left((1-\alpha)K + \alpha K'\right)\big)^{\frac{1}{n}} \geq (1-\alpha)\big(\mathcal{H}^n\left(K\right)\big)^{\frac{1}{n}} + \alpha\big(\mathcal{H}^n\left(K'\right)\big)^{\frac{1}{n}}.$$

In other words, the map $\mathcal{C}_b \to \mathbb{R}, K \mapsto \big(\mathcal{H}^n\left(K\right)\big)^{\frac{1}{n}}$ is concave. Then for fixed convex bodies $K, K' \subset V$ the map $[0,1] \to \mathbb{R}, \alpha \mapsto \big(\mathcal{H}^n\left((1-\alpha)K + \alpha K'\right)\big)^{\frac{1}{n}}$ is a concave function. Define the function

$$\Phi\colon [0,1] \to \mathbb{R}, \alpha \mapsto \big(\mathcal{H}^n\left((1-\alpha)K + \alpha K'\right)\big)^{\frac{1}{n}} - (1-\alpha)\big(\mathcal{H}^n\left(K\right)\big)^{\frac{1}{n}} - \alpha\big(\mathcal{H}^n\left(K'\right)\big)^{\frac{1}{n}}$$

which is also concave since the additional terms are linear in α. Note that $\Phi(0) = \big(\mathcal{H}^n\left(K\right)\big)^{\frac{1}{n}} - \big(\mathcal{H}^n\left(K\right)\big)^{\frac{1}{n}} = 0$ and similarly $\Phi(1) = 0$. Note further that Φ is a polynomial function in the variable α due to expression (1.2.18) and is thus differentiable on $(0,1)$ and the one-sided limits for the difference quotient at $\alpha = 0$ and $\alpha = 1$ exist. The concavity and continuity of Φ imply that its (right hand) derivative at $\alpha = 0$ must be non-negative. We will shortly see that this implies the claim but first let us calculate the derivative of $\mathcal{H}^n\left((1-\alpha)K + \alpha K'\right)$ at $\alpha = 0$. Equation (1.2.18) yields

$$\begin{aligned}
&\left.\frac{\mathrm{d}}{\mathrm{d}\alpha}\right|_{\alpha=0} \Big(\mathcal{H}^n\left((1-\alpha)K + \alpha K'\right)\Big) \\
&= \left.\frac{\mathrm{d}}{\mathrm{d}\alpha}\right|_{\alpha=0} \left(\sum_{j=0}^{n} \binom{n}{j} \mathrm{V}\left(K[n-j], K'[j]\right)(1-\alpha)^{n-j}\alpha^j\right) \\
&= \sum_{j=0}^{n} \binom{n}{j} \mathrm{V}\left(K[n-j], K'[j]\right) \left.\frac{\mathrm{d}}{\mathrm{d}\alpha}\right|_{\alpha=0} \Big((1-\alpha)^{n-j}\alpha^j\Big)
\end{aligned}$$

$$
\begin{aligned}
&= \mathrm{V}\left(K[n], K'[0]\right) \frac{\mathrm{d}}{\mathrm{d}\alpha}\Big|_{\alpha=0} (1-\alpha)^n + n\,\mathrm{V}\left(K[n-1], K'[1]\right) \frac{\mathrm{d}}{\mathrm{d}\alpha}\Big|_{\alpha=0} \Big((1-\alpha)^{n-1}\alpha\Big) \\
&\quad + \sum_{j=2}^{n} \binom{n}{j} \mathrm{V}\left(K[n-j], K'[j]\right) \frac{\mathrm{d}}{\mathrm{d}\alpha}\Big|_{\alpha=0} \Big((1-\alpha)^{n-j}\alpha^j\Big) \\
&= -\, n\mathcal{H}^n\left(K\right) + n\,\mathrm{V}\left(K[n-1], K'[1]\right) \\
&\quad + \sum_{j=2}^{n} \binom{n}{j} \mathrm{V}\left(K[n-j], K'[j]\right) \Big(j\alpha^{j-1}(1-\alpha)^{n-j} - (n-j)\alpha^j(1-\alpha)^{n-j-1}\Big)\Big|_{\alpha=0} \\
&= n\Big(\mathrm{V}\left(K[n-1], K'[1]\right) - \mathcal{H}^n\left(K\right)\Big).
\end{aligned}
\tag{1.2.20}
$$

The last equality uses the fact that all summands with index $j \geq 2$ contain a power of α which vanishes at 0. Proceeding, one gets

$$
\begin{aligned}
\frac{\mathrm{d}\Phi}{\mathrm{d}\alpha}\Big|_{\alpha=0} &= \frac{\mathrm{d}}{\mathrm{d}\alpha}\Big|_{\alpha=0} \Big[\big(\mathcal{H}^n\left((1-\alpha)K + \alpha K'\right)\big)^{\frac{1}{n}} - (1-\alpha)\big(\mathcal{H}^n\left(K\right)\big)^{\frac{1}{n}} - \alpha\big(\mathcal{H}^n\left(K'\right)\big)^{\frac{1}{n}}\Big] \\
&= \frac{1}{n}\big(\mathcal{H}^n\left(K\right)\big)^{\frac{1}{n}-1} \frac{\mathrm{d}}{\mathrm{d}\alpha}\Big|_{\alpha=0} \Big[\mathcal{H}^n\left((1-\alpha)K + \alpha K'\right)\Big] + \big(\mathcal{H}^n\left(K\right)\big)^{\frac{1}{n}} - \big(\mathcal{H}^n\left(K'\right)\big)^{\frac{1}{n}}.
\end{aligned}
$$

Using (1.2.20), the last statement can be rewritten as

$$
\begin{aligned}
\frac{\mathrm{d}\Phi}{\mathrm{d}\alpha}\Big|_{\alpha=0} &= \big(\mathcal{H}^n\left(K\right)\big)^{\frac{1}{n}-1}\Big[\mathrm{V}\left(K[n-1], K'[1]\right) - \mathcal{H}^n\left(K\right)\Big] + \big(\mathcal{H}^n\left(K\right)\big)^{\frac{1}{n}} - \big(\mathcal{H}^n\left(K'\right)\big)^{\frac{1}{n}} \\
&= \big(\mathcal{H}^n\left(K\right)\big)^{-\frac{n-1}{n}} \mathrm{V}\left(K[n-1], K'[1]\right) - \big(\mathcal{H}^n\left(K'\right)\big)^{\frac{1}{n}} \\
&= \big(\mathcal{H}^n\left(K\right)\big)^{-\frac{n-1}{n}} \Big\{\mathrm{V}\left(K[n-1], K'[1]\right) - \big(\mathcal{H}^n\left(K\right)\big)^{\frac{n-1}{n}} \big(\mathcal{H}^n\left(K'\right)\big)^{\frac{1}{n}}\Big\}.
\end{aligned}
$$

Thus, the non-negativity of $\frac{\mathrm{d}\Phi}{\mathrm{d}\alpha}\big|_{\alpha=0}$ implies the claim. □

1.2.5 An explicit formula for the mixed volume of a polytope and a convex set

There is an intuitive geometrical way of calculating the Minkowski sum of a polytope A and a convex body K. First, consider the set $K' := K - k$ where k is an interior point of K. Secondly, attach the set K' to each vertex of A and take the convex hull (see Figure 1.2 (a)). Lastly, translate by the vector k, that is, $A + K = \big(A + (K-k)\big) + k$.

Lemma 1.2.13

Suppose $A_N = A$ is a polytope and K a convex body in V and k an interior point of K. Their

Minkowski sum is given by

$$A + K = \operatorname{conv}\left(\bigcup_{i=1}^{N} (a_i + K')\right) + k$$

where $K' := K - k$.

PROOF: Without loss of generality we may assume the origin is an interior point of K and $k = 0$. For the sake of brevity let $C := \operatorname{conv}\left(\bigcup_{i=1}^{N} (a_i + K)\right)$. Clearly, $C \subset A + K$, for if $c \in C$ then $c = \sum_{l=1}^{n+1} \lambda_l (a_{i_l} + k_l)$ where $\sum_{l=1}^{n+1} \lambda_l = 1$, $k_l \in K$ and $a_{i_l} \in \{a_1, a_2, \ldots, a_N\}$ for $l = 1, 2, \ldots, N$. Since both A and K are convex $c = \sum_{l=1}^{n+1} \lambda_l a_{i_l} + \sum_{l=1}^{n+1} \lambda_l k_l \in A + K$.

Conversely, let $c \in A + K$. Since each point in the polytope A is a convex combination of the vertices $a_1, a_2, \ldots, a_N$, it follows that

$$c = \sum_{i=1}^{N} \lambda_i a_i + k = \sum_{i=1}^{N} \lambda_i (a_i + k) \in C$$

where $\sum_{i=1}^{N} \lambda_i = 1$. □

As a motivation for the next result consider a polytope $A \subset V$ with N facets F_i. Recall that facets are the $(n-1)$-dimensional extreme subsets of A. Without loss of generality, suppose A contains the origin $0 \in V$ as an interior point. An intuitive way of calculating its volume is the following geometric sequence of steps. First, one triangulates the polytope by connecting each of its facets with the origin which creates N pyramids $P_i = \operatorname{conv}(F_i \cup \{0\})$. Secondly, one adds up the volume of each pyramid. Lastly, one recalls that the volume of a pyramid is simply the $(n-1)$-dimensional volume of the "base" times its "height" divided by the dimension n. (This can be proven by using Cavalieri's principle, see e.g. [For09, §5, Satz 3, p. 49].) To put this into more mathematical terms, intuitively, the volume of A is

$$\mathcal{H}^n(A) = \sum_{i=1}^{N} \mathcal{H}^n(P_i) = \frac{1}{n} \sum_{i=1}^{N} \mathcal{H}^{n-1}(F_i) h_A\big(f^i\big)$$

where f^i denotes the outer unit normal (see p. 36) of the facet F_i and h_A is the support function of A. Due to Proposition 1.2.6 the value $h_A(f^i)$ is the distance of the affine subspace containing the facet F_i to the origin (and so the height of the pyramid P_i).

With the help of Corollary 1.2.11 the mixed volume of a polytope and a convex body can be calculated geometrically. This leads to a formula very similar to the one which we just motivated.

Lemma 1.2.14 ([BF71, §29, Formula (3), p. 41ff])
Let A be a polytope with facets $F_1, F_2, \dots, F_N$, K a convex body in V and k an arbitrary interior point of K. Suppose f^i is the outer unit normal to the facet F_i and h_K is the support function of K. The mixed volume of $\underbrace{A, A, \dots, A}_{(n-1)-\text{times}}$ and K is given by

$$\mathrm{V}\left(A[n-1], K\right) = \frac{1}{n}\sum_{i=1}^{N} \mathcal{H}^{n-1}(F_i) h_{K-k}\big(f^i\big). \tag{1.2.21}$$

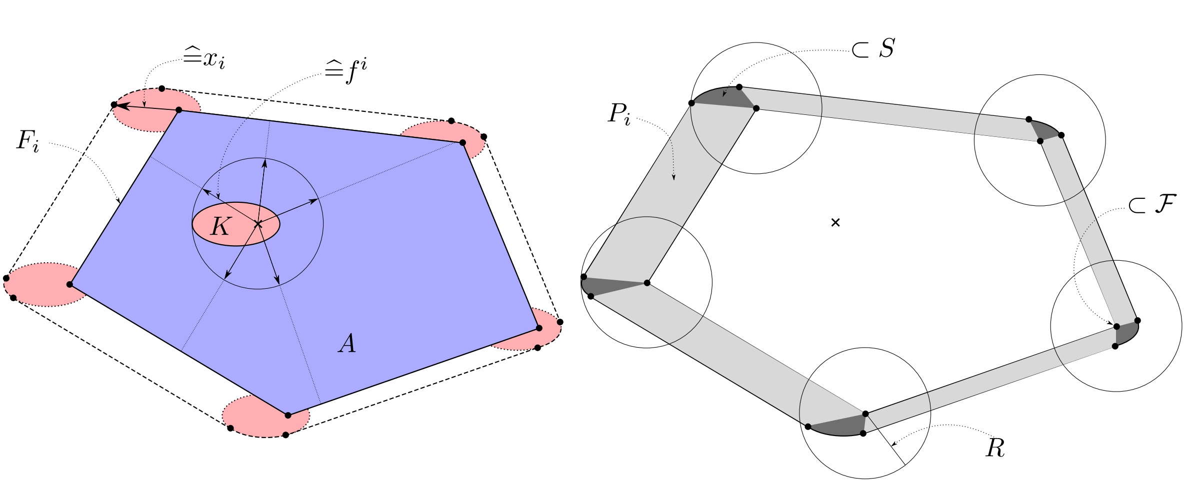
Proof: Since mixed volume is invariant under translation of any of its arguments, the left hand side $\mathrm{V}\left(A[n-1], K\right)$ equals $\mathrm{V}\left(A[n-1], K-k\right)$. Without loss of generality, we thereby may assume K contains the origin of V and $k = 0$. Further suppose $n \geq 2$, for if $n = 1$ then A and $K = [k_1, k_2]$ are segments in the one-dimensional vector space V. In addition, $N = 2$ because the number of vertices is minimal by definition of a polytope and, in fact, the facets of A coincide with the vertices which are the endpoints of the segment. The value $h_K(f^i)$ gives the distance of $0 \in K$ to its endpoints (Proposition 1.2.6) and because the 0-dimensional Hausdorff measure counts the number of points in a set, $\mathcal{H}^0(F_i) = 1$. Thus, $\frac{1}{1}\sum_{i=1}^{2} \mathcal{H}^0(F_i) h_K(f^i) = \sum_{i=1}^{2} h_K(f^i) = \|k_1\|_V + \|k_2\|_V = \|k_1 - k_2\|_V = \mathcal{H}^1(K) = \mathrm{V}\left(A[0], K\right)$. The third equality holds because the segment $[k_1, k_2]$ contains the origin, so $k_1 = -|k_1| e_1$ and $k_2 = |k_2| e_1$ point in opposite directions.
By Corollary 1.2.11 the mixed volume in the assertation can be calculated through

$$\mathrm{V}\left(A[n-1], K\right) = \frac{1}{n}\lim_{t \to 0} \frac{\mathcal{H}^n\left(A + tK\right) - \mathcal{H}^n\left(A\right)}{t}.$$

So, a further examination of $\mathcal{H}^n\left(A + tK\right)$ is needed. We will show that the set $A + tK$ can be subdivided into several smaller pieces whose volumes are easier to calculate. According to Proposition 1.2.6 the hyperplane $H_i := \{y \in V \mid f^i(y) = h_K(f^i)\}$ supports K, whereby there is $x_i \in H_i \cap K$ for each $i = 1, 2, \dots, N$. Define the set $P_i := \{y + s x_i \mid y \in F_i, 0 \leq s \leq t\} = F_i + [0, t x_i]$. This set describes a prism with $(n-1)$-dimensional base F_i which is translated into direction x_i (see Figure 1.2 (b)). Its "thickness" or height is

$$h := h_{A+tK}(f^i) - h_A(f^i) = h_A(f^i) + t h_K(f^i) - h_A(f^i) = t h_K(f^i)$$

where we used Proposition 1.2.6 twice to calculate the distances of the origin to the supporting hyperplanes for $A + tK$ and A defined by the normal f^i. The hyperplanes $H_s(f^i) = \{y \in V \mid f^i(y) = s\}$ are parallel to the facet F_i. Further, the intersection of a prism P_i with these hyperplanes is either



(a) Minkowski sum $A + K$

(b) Dividing the Minkowski sum into prisms P_i (light gray) and the remainder S (dark gray)

Figure 1.2: Calculation of the mixed volume $\mathrm{V}(A[n-1], K)$

empty or a translation of the facet F_i into direction x_i. In particular,

$$P_i \cap H_s(f^i) = \begin{cases} F_i + v_s & , s \in [h_A(f^i), h_{A+tK}(f^i)] \\ \varnothing & , s \notin [h_A(f^i), h_{A+tK}(f^i)]. \end{cases}$$

where $v_s \in [0, tx_i]$ is a vector in direction x_i (of length $\frac{s - h_A(f^i)}{h_K(f^i)} \|x_i\|_V$). Then by Cavalieri's principle ([For09, §5, Satz 3, p. 49]) we have

$$\begin{aligned} \mathcal{H}^n(P_i) &= \int_{\mathbb{R}} \mathcal{H}^{n-1}(P_i \cap H_s(f^i)) \, \mathrm{d}s = \int_{h_A(f^i)}^{h_{A+tK}(f^i)} \mathcal{H}^{n-1}(F_i + v_s) \, \mathrm{d}s \\ &= \int_{h_A(f^i)}^{h_A(f^i)+h} \mathcal{H}^{n-1}(F_i) \, \mathrm{d}s = h\mathcal{H}^{n-1}(F_i) \\ &= t\mathcal{H}^{n-1}(F_i) h_K(f^i). \end{aligned}$$

Each of these N prisms P_i is contained in the sum $A + tK$ (by definition of the Minkowski sum) but not in A (by construction). Therefore,

$$A + tK = A \cup \bigcup_{i=1}^{N} P_i \cup S$$

where $S := (A + tK) \setminus \left(A \cup \bigcup_{i=1}^{N} P_i \right)$ is the remaining part of the Minkowski sum "close to" the $(n-2)$-dimensional faces of A (dark gray in Figure 1.2 (b)). Note that this decomposition of $A + tK$

is not pairwise disjoint but any intersection of the sets A, P_i and S is lower-dimensional and thus a set of n-dimensional Hausdorff measure zero. Therefore, we conclude

$$\begin{aligned}
\mathrm{V}\left(A[n-1],K\right) &= \frac{1}{n}\lim_{t\to 0}\frac{\mathcal{H}^n\left(A+tK\right)-\mathcal{H}^n\left(A\right)}{t}\\
&= \frac{1}{n}\lim_{t\to 0}\frac{\mathcal{H}^n\left(A\right)+\sum_{i=1}^N\mathcal{H}^n\left(P_i\right)+\mathcal{H}^n\left(S\right)-\mathcal{H}^n\left(A\right)}{t}\\
&= \frac{1}{n}\lim_{t\to 0}\frac{\sum_{i=1}^N\mathcal{H}^{n-1}(F_i)th_K(f^i)+\mathcal{H}^n\left(S\right)}{t}\\
&= \frac{1}{n}\sum_{i=1}^N\mathcal{H}^{n-1}(F_i)h_K(f^i)+\frac{1}{n}\lim_{t\to 0}\frac{\mathcal{H}^n\left(S\right)}{t}.
\end{aligned}$$

We need to further investigate the term $\lim_{t\to 0}\frac{\mathcal{H}^n(S)}{t}$.

The intersection of a facet F_i with another facet F_j is either empty or a face of A of dimension $j\le n-2$ ([Brø83, Theorem 5.9, pp. 30+33 and Theorem 10.4, p. 65]). Recall that a face of a polytope again is a polytope and therefore compact by Lemma 1.2.3. Then the set

$$\mathcal{F}:=\mathcal{F}^{n-2}(A):=\bigcup_{\substack{i,j=1\\ i\neq j}}^N\left(F_i\cap F_j\right)$$

is the union of all those faces of A which are of dimension $j\le n-2$. (In the two-dimensional drawing Figure 1.2 the set $\mathcal{F}$ is represented by the "vertices" which actually consist of $(n-2)$-dimensional faces of A.) The set $\mathcal{F}$ is compact since it is the finite union of compact sets.

Define the radius $R:=\sup_{\|f\|_{V^*}=1}h_K(f)$ and recall from Proposition 1.2.6 that for each unit length $f\in V^*$ the hyperplane $H_{h_K(f)}(f)=\{v\in V\mid f(v)=h_K(f)\}$ supports K, that is, $H_{h_K(f)}(f)\cap K\neq\varnothing$ and the value $h_K(f)$ is the distance of $0\in K$ to this hyperplane. Geometrically, we may think of

$$R=\sup_{\|f\|_{V^*}=1}h_K(f)=\sup\left\{\operatorname{dist}(0,H_{h_K(f)}(f))\;\middle|\;f\in V^*,\|f\|_{V^*}=1\right\}$$

as the supremal distance of $0\in K$ to the boundary of K or, in other words, the radius of the smallest ball which contains K (Figure 1.2). That is,

$$A\cup\bigcup_{i=1}^N P_i\cup S=A+tK\subset A+tB_R(0)=\left(\operatorname{int}(A)\cup\bigcup_{i=1}^N F_i\cup\mathcal{F}\right)+tB_R(0).$$

However, since by definition $S \not\subset \left(A \cup \bigcup_{i=1}^{N} P_i\right)$, we know that actually

$$S \subset \mathcal{F} + tB_R(0) = \bigcup_{c\in\mathcal{F}} B_{tR}(c).$$

Since the closure $\overline{S}$ is compact (because its bounded as a subset of the bounded convex body $A + tK$) we may take a subcover of balls of radius tR centered at finitely many, say $M = M(K)$, points $c_i \in \mathcal{F}$. Therefore, the volume of S calculates to

$$\mathcal{H}^n\left(S\right) = \mathcal{H}^n\left(\overline{S}\right) \leq \sum_{i=1}^{M} \mathcal{H}^n\left(B_{tR}(c_i)\right) = M\mathcal{H}^n\left(B_1(0)\right) R^n t^n = Ct^n$$

where $C = C(A, K, n) > 0$ is a constant. Since $n \geq 2$, we conclude

$$0 \leq \lim_{t\to 0} \frac{\mathcal{H}^n\left(S\right)}{t} \leq C \lim_{t\to 0} t^{n-1} = 0$$

which finishes the proof. □

Note that if $K = A$ the previous result produces the formula for the volume of a polytope which was motivated earlier. Thus, a posteriori, we have also given a (rather complicated) proof of the usual volume formula for polytopes.

For the next chapter we need to prove the following result about polytopes in a two-dimensional space which intuitively holds true. Indeed, the proof is quite straightforward if one unravels all the tedious definitions that will be made.

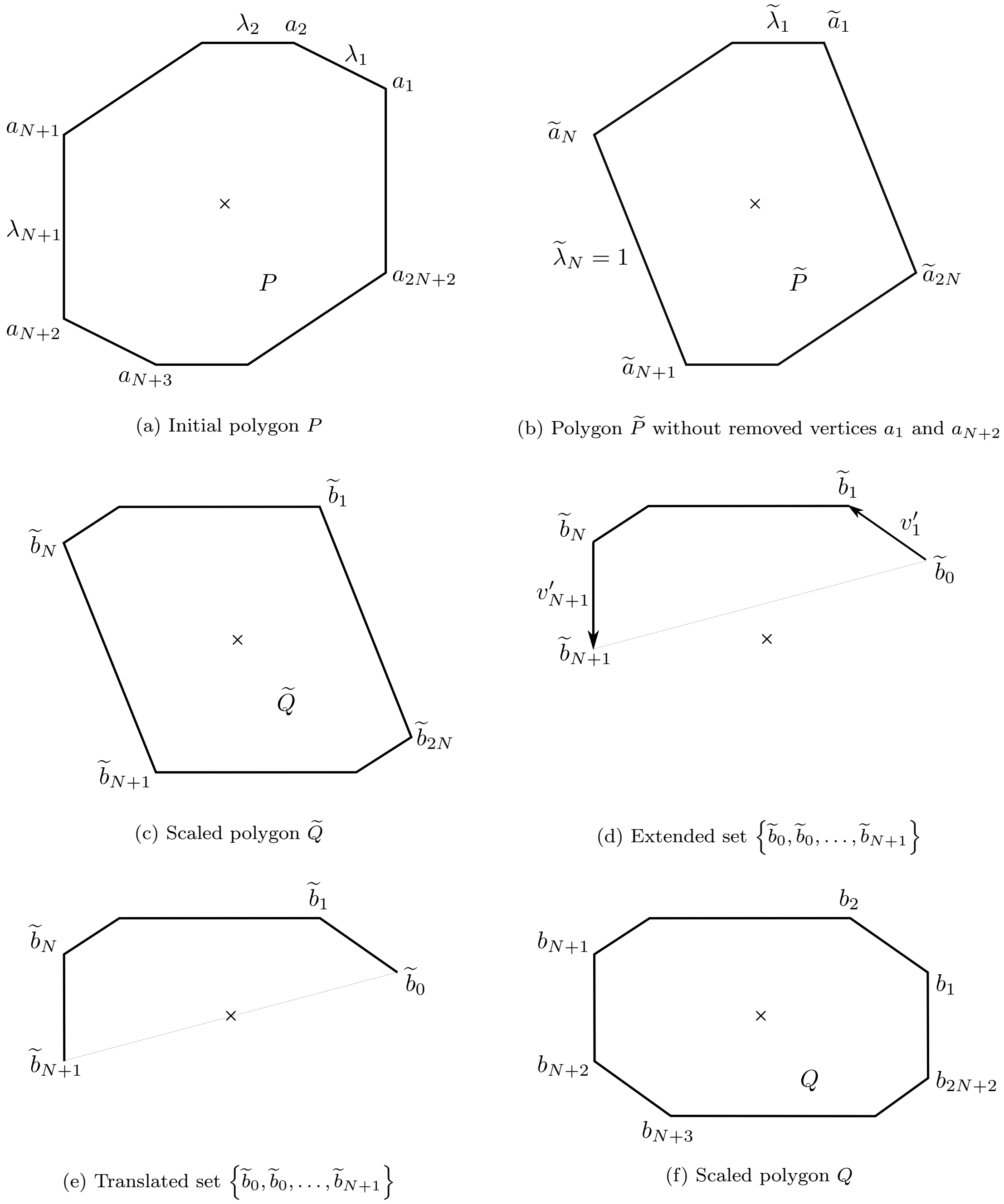
Lemma 1.2.15

Let $P = [a_1 a_2 \dots a_{2N}]$ be a symmetric polygon in a two-dimensional normed space V with $2N$ vertices. Further, let $\lambda_1, \lambda_2, \dots, \lambda_N$ be arbitrary positive real numbers. Then there exists a symmetric polygon $Q = [b_1 b_2 \dots b_{2N}]$ with $2N$ vertices such that

$$b_{i+1} - b_i = \lambda_i(a_{i+1} - a_i)$$

for all $i = 1, 2, \dots, N$, that is, the ith edge of P is scaled by the factor λ_i and therefore the ith edges of P and Q are parallel.

Proof: The polygon Q can be explicitly constructed from P. We will prove this lemma by induction on N. When $N = 1$, the statement is trivial because we can simply choose $b_1 := \lambda_1 a_1$ and $b_2 := -b_1$. Suppose the assertion holds for an $N \in \mathbb{N}$ and let $P = [a_1 a_2 \dots a_{2(N+1)}]$ be a symmetric polygon with



(a) Initial polygon P

(b) Polygon $\widetilde{P}$ without removed vertices a_1 and a_{N+2}

(c) Scaled polygon $\widetilde{Q}$

(d) Extended set $\left\{\widetilde{b}_0, \widetilde{b}_0, \dots, \widetilde{b}_{N+1}\right\}$

(e) Translated set $\left\{\widetilde{b}_0, \widetilde{b}_0, \dots, \widetilde{b}_{N+1}\right\}$

(f) Scaled polygon Q

Figure 1.3: Constructing the scaled polygon Q from P for $2(N+1)$ vertices

$2(N+1)$ vertices. For $i = 1, 2, \ldots, N+1$ define the vectors $v_i := a_{i+1} - a_i$ corresponding to the edges of P and $v_i' := \lambda_i v_i$ which will become the edges of the polygon Q. To be able to apply the inductive hypothesis we need to construct a symmetric polygon with $2N$ vertices (Figure 1.3 (a)). We will do this by removing the vertices a_1 and $a_{N+2} = -a_1$, that is, define the symmetric $2N$-gon $\widetilde{P} = [\widetilde{a}_1 \widetilde{a}_2 \ldots \widetilde{a}_{2N}]$ where

$$\widetilde{a}_i := \begin{cases} a_{i+1} & i = 1, 2, \ldots, N \\ a_{i+2} & i = N+1, N+2, \ldots, 2N \end{cases}.$$

Further, define the new scaling coefficients

$$\widetilde{\lambda}_i := \begin{cases} \lambda_{i+1} & i = 1, 2, \ldots, N-1 \\ 1 & i = N \end{cases},$$

that is, the edge $\widetilde{v}_N := \widetilde{a}_{N+1} - \widetilde{a}_N$ will not be scaled (Figure 1.3 (b)). By the inductive hypothesis there exists a symmetric $2N$-gon $\widetilde{Q} = [\widetilde{b}_1 \widetilde{b}_2 \ldots \widetilde{b}_{2N}]$ such that

$$\widetilde{b}_{i+1} - \widetilde{b}_i = \widetilde{\lambda}_i (\widetilde{a}_{i+1} - \widetilde{a}_i)$$

for all $i = 1, 2, \ldots, N$ (Figure 1.3 (c)). Consider only the set $\left\{\widetilde{b}_1, \widetilde{b}_2, \ldots, \widetilde{b}_N\right\}$ of the first N vertices. For notational convenience let us forget about the remaining vertices and (re-)define two new vertices by going along the initial scaled edges, namely, $\widetilde{b}_0 := \widetilde{b}_1 - v_1'$ and $\widetilde{b}_{N+1} := \widetilde{b}_N + v_{N+1}'$ (Figure 1.3 (d)). Translate the extended vertex set $\left\{\widetilde{b}_0, \widetilde{b}_0, \ldots, \widetilde{b}_{N+1}\right\}$ affinely such that $\widetilde{b}_{N+1} = -\widetilde{b}_0$ and denote the points by the same name (Figure 1.3 (e)). In fact, the map $T \colon V \to V, v \mapsto v - \frac{1}{2}\left(\widetilde{b}_{N+1} + \widetilde{b}_0\right)$ is such an affine translation. Finally, we define the vertices

$$b_i := \begin{cases} \widetilde{b}_{i-1} & i = 1, 2, \ldots, N+1 \\ -\widetilde{b}_{i-1-(N+1)} & i = N+2, N+3, \ldots, 2N+2 \end{cases}.$$

The $2(N+1)$-gon $Q = [b_1 b_2 \ldots b_{2N+2}]$ is symmetric by construction (Figure 1.3 (f)). Unwinding the definitions, we compute

$$b_{i+1} - b_i = \widetilde{b}_i - \widetilde{b}_{i-1} = \widetilde{\lambda}_{i-1}(\widetilde{a}_i - \widetilde{a}_{i-1}) = \lambda_i (a_{i+1} - a_i)$$

for $i = 2, 3, \dots, N+1$. Moreover, for $i = 1$

$$b_2 - b_1 = \widetilde{b}_1 - \widetilde{b}_0 = v_1' = \lambda_1 v_1 = \lambda_1(a_2 - a_1). \qquad \square$$

1.2.6 Convex and linear functions on convex sets

This section ends with some results on the action of linear and convex functions on convex sets.

Lemma 1.2.16

Let V_1 and V_2 be two finite-dimensional normed spaces and $L\colon V_1 \to V_2$ a linear map.

(i) If $K_1 \subset V_1$ is a convex body then $L(K_1) \subset V_2$ is a convex body.

(ii) If $K_2 \subset \operatorname{im}(L) \subset V_2$ is a convex body then the preimage $L^{-1}(K_2) \subset V_1$ is a convex set. If further L is injective then $L^{-1}(K_2)$ is a convex body.

(iii) Suppose $P_2 \subset \operatorname{im}(L) \subset V_2$ is a polytope containing the origin given by its half-space representation (Theorem 1.2.7)

$$P_2 = \bigcap_{i=1}^{M} \{ y \in V_2 \mid F^i(y) \leq 1 \}$$

for $F^i \in V_2^*$. Then the preimage $P_1 := L^{-1}(P_2) \subset V_1$ is a polytope and its half-space representation is given by

$$P_1 = \bigcap_{i=1}^{M} \{ x \in V_1 \mid f^i(x) \leq 1 \}$$

where $f^i := F^i \circ L \in V_1^*$.

(iv) If $K_2 \subset V_2$ is symmetric then the preimage $L^{-1}(K_2) \subset V_1$ is a symmetric set.

Proof: For $i = 1, 2$ let $x_i \in K_1$ and $y_i \in K_2$ such that $y_i = Lx_i$ and let $t \in [0, 1]$. Then by the linearity of L

$$ty_1 + (1-t)y_2 = tLx_1 + (1-t)Lx_2 = L(tx_1 + (1-t)x_2).$$

If K_1 is convex this equality implies the convexity of $L(K_1)$ if read from right to left and conversely, if read from left to right it implies the convexity of $L^{-1}(K_2)$ if K_2 is convex. If x_0 is an interior point of K_1 there is $\varepsilon > 0$ such that $B_1 := B_{\varepsilon \|L\|^{-1}_{\mathcal{L}(V_1,V_2)}}(x_0) \subset K_1$. Let x be an arbitrary point in B_1. Then

$Lx \in L(K_1)$ but also $\|Lx - Lx_0\|_{V_2} \leq \|L\|_{\mathcal{L}(V_1,V_2)} \|x - x_0\|_{V_2} < \|L\|_{\mathcal{L}(V_1,V_2)} \varepsilon \|L\|^{-1}_{\mathcal{L}(V_1,V_2)} = \varepsilon$, that is, $B_\varepsilon(Lx_0) \subset L(K_1)$. In other words, $L(K_1)$ has non-empty interior. Since a linear map is continuous, the image of a compact set is compact. Therefore, $L(K_1)$ is a convex body if K_1 is.

For part (ii) note that $\widetilde{L}\colon V_1 \to \mathrm{im}\,(L), v \mapsto Lv$ is bijective and $L^{-1}(K_2) = \widetilde{L}^{-1}(K_2 \cap \mathrm{im}\,(L))$. Thus, we just interchange the roles of K_1, V_1 and K_2, V_2, respectively, and apply (i) to the linear map $\widetilde{L}^{-1}\colon V_2 \to V_1$.

For part (iii) denote $Q := \bigcap_{i=1}^{M} \{x \in V_1 \mid f^i(x) \leq 1\}$ and let $v \in P_1$. Then $Lv \in P_2$, that is, $1 \geq F^i(Lv) = f^i(v)$ for $i = 1, 2, \ldots, M$. Thus, $v \in Q$. Conversely, if $v \in Q$ then $f^i(v) = F^i(Lv) \leq 1$ for $i = 1, 2, \ldots, M$, that is, $Lv \in P_2$. Hence, $v \in P_1$.

For part (iv) suppose K_2 is symmetric and let $x \in L^{-1}(K_2)$, that is, $Lx \in K_2$. Since K_2 is symmetric, we know that $L(-x) = -(Lx) \in K_2$ or $-x \in L^{-1}(K_2)$. So, $L^{-1}(K_2)$ is symmetric. □

Let us conclude this section with a result on the maxima of convex functions on polytopes. Suppose $A \subset V$ is a convex set. A map $F\colon A \to \mathbb{R} \cup \{-\infty, +\infty\}$ is called a ***convex function*** if

$$F(\lambda v + (1-\lambda)w) \leq \lambda F(v) + (1-\lambda)F(w)$$

whenever $\lambda \in [0,1]$ and $v, w \in A$. We call a convex function F ***proper*** if it nowhere takes the value $-\infty$ and is not identically $+\infty$.

An easy induction proves that the inequality above holds for arbitrary finite convex combinations.

Lemma 1.2.17

Let $A \in V$ be a convex set, $v_1, v_2, \ldots, v_N \in A$, $\lambda_1, \lambda_2, \ldots, \lambda_N \in [0,1]$ with $\sum_{i=1}^{N} \lambda_i = 1$ and $F\colon A \to \mathbb{R}$ a convex function. Then

$$F\left(\sum_{i=1}^{N} \lambda_i v_i\right) \leq \sum_{i=1}^{N} \lambda_i F(v_i).$$

Proof: For $N = 1$ the claim is trivial and for $N = 2$ it follows from the definition of a convex function. Assume the assertion holds for $N-1$. Note that $\sum_{i=1}^{N-1} \lambda_i = 1 - \lambda_N$ or $\sum_{i=1}^{N-1} \frac{\lambda_i}{1-\lambda_N} = 1$. Define $w := \sum_{i=1}^{N-1} \frac{\lambda_i}{1-\lambda_N} v_i \in V$. Then

$$\begin{aligned} F\left(\sum_{i=1}^{N} \lambda_i v_i\right) &= F\left(\sum_{i=1}^{N-1} \lambda_i v_i + \lambda_N\right) = F((1-\lambda_N)w + \lambda v_N v_N) \\ &\leq (1-\lambda_N)F(w) + \lambda F(v_N) \\ &= (1-\lambda_N)F\left(\sum_{i=1}^{N-1} \frac{\lambda_i}{1-\lambda_N} v_i\right) + \lambda F(v_N) \end{aligned}$$

$$\leq (1-\lambda_N)\sum_{i=1}^{N-1}\frac{\lambda_i}{1-\lambda_N}F(v_i)+\lambda F(v_N)$$
$$=\sum_{i=1}^{N}\lambda_i F(v_i)$$

where we used the definition of a convex function for the first inequality and the inductive hypothesis for the second. □

The following important maximum principle for convex functions holds.

Proposition 1.2.18

Let $P=[a_1a_2\dots a_N]$ be a polytope in V and $F\colon V\to\mathbb{R}$ a convex function which is bounded above on P. Then the supremum of F on P is attained at one of its vertices, that is,

$$\sup_{x\in P}F(x)=\sup_{i=1,2,\dots,N}F(a_i)=\max_{i=1,2,\dots,N}F(a_i).$$

Proof: The second identity follows from the fact that $\{a_1,a_2,\dots,a_N\}$ is a finite set. It is immediate that $\sup_{i=1,2,\dots,N}F(a_i)\leq\sup_{x\in P}F(x)$ since $\{a_1,a_2,\dots,a_N\}\subset\operatorname{conv}(a_1,a_2,\dots,a_N)=P$. Conversely, let $v=\sum_{i=1}^N\lambda_ia_i\in P$ where $\sum_{i=1}^N\lambda_i=1$. Then the convexity of F yields

$$F(v)=F\left(\sum_{i=1}^N\lambda_ia_i\right)\leq\sum_{i=1}^N\lambda_iF(a_i)\leq\sum_{i=1}^N\lambda_i\max_{k=1,2,\dots,N}F(a_k)=\max_{k=1,2,\dots,N}F(a_k)$$

since v was arbitrary we get $\sup_{x\in P}F(x)\leq\max_{k=1,2,\dots,N}F(a_k)$. □

In fact, this result holds true if P is merely convex and not necessarily a polytope but then the supremum is attained on the set of extreme points (the extreme subsets that consist only of one point), see [Roc70, Corollary 32.3.1, p. 344]. For the application in the next chapter only the special case above is needed.

1.3 Differential geometry

In this section we introduce basic concepts from differential geometry, such as the notion of a smooth manifold and its tangent space and tangent bundle. We describe the differential of a smooth map and examine smooth immersions and images of such maps. Finally, we consider Finsler metrics. In this section and for the rest of this thesis we will use the Einstein summation convention, which means that – if not explicitly stated otherwise – we sum over repeated upper and lower indices. We use the convention that Latin indices run from 1 to n and Greek indices run from 1 to m. The presentation given here is borrowed from [Ove13, Chapters 1.3 - 1.4, pp. 28 - 41, Chapter 2] which in turn is based on the book of Lee [Lee13]. The source for the concepts of Finsler geometry is the textbook [BCS00] by Bao et al.

1.3.1 Smooth manifolds

Suppose $\mathcal{M}$ is a topological space. We call $\mathcal{M}$ a ***topological manifold of dimension m*** or a ***topological m-manifold*** if it fulfils the following three properties:

- $\mathcal{M}$ is a ***Hausdorff space***: for every pair of distinct points $p, q \in \mathcal{M}$, there are disjoint open subsets $U, V \subset \mathcal{M}$ such that $p \in U$ and $q \in V$.
- $\mathcal{M}$ is ***second-countable***: there exists a countable basis for the topology of $\mathcal{M}$.
- $\mathcal{M}$ is ***locally Euclidean of dimension m***: for each point $p \in \mathcal{M}$ there is an open subset $U \subset \mathcal{M}$ containing p, an open subset $\widehat{U} \subset \mathbb{R}^m$ and a homeomorphism $\varphi\colon U \to \widehat{U}$.

A ***(local) coordinate chart*** is a pair (U, φ) where U is an open subset of $\mathcal{M}$ and $\varphi\colon U \to \widehat{U}$ is a homeomorphism onto an open subset $\widehat{U} = \varphi(U) \subset \mathbb{R}^m$. The set U is called a ***coordinate neighbourhood*** and the map φ a ***(local) coordinate map***. The component functions $(u^1, u^2, \dots, u^m)$ of φ, defined by $\varphi(p) = (u^1(p), u^2(p), \dots, u^m(p)))$, are called ***(local) coordinates***. Sometimes we write $\widehat{p} = (u^1(p), u^2(p), \dots, u^m(p))$ for the local coordinates of a point $p \in \mathcal{M}$ and we use any of the notations $(U, \varphi) = (U, (u^1, u^2, \dots, u^m)) = (U, (u^\alpha))$ for a coordinate chart (see [Lee13, pp. 2–4]). Another way to emphasise the dimension $\dim(\mathcal{M})$ of a manifold is to write it as an upper index, e.g. $\mathcal{M}^m$.

To make sense of derivatives of functions on a manifold or maps between manifolds we need to introduce an additional structure which turns a topological into a smooth manifold. Consider two coordinate charts (U, φ) and (V, ψ) of a topological manifold $\mathcal{M}$ where $U \cap V \neq \varnothing$. The ***transition***

map between φ and ψ is the mapping

$$\psi \circ \varphi^{-1} \colon \varphi(U \cap V) \subset \mathbb{R}^m \to \psi(U \cap V) \subset \mathbb{R}^m.$$

A ***smooth atlas $\mathcal{A}$*** is the union of all those coordinate charts of $\mathcal{M}$ such that the union of the coordinate neighbourhoods cover $\mathcal{M}$ and each transition map is a smooth map from $\mathbb{R}^m$ to $\mathbb{R}^m$, that is, $\mathcal{A}$ fulfils the following two properties:

- $\mathcal{M} = \bigcup_{(U,\varphi)\in\mathcal{A}} U$,
- $\psi\circ\varphi^{-1} \colon \varphi(U\cap V) \to \psi(U\cap V)$ is of class $C^\infty(\mathbb{R}^m, \mathbb{R}^m)$ for all coordinate charts $(U,\varphi), (V,\psi) \in \mathcal{A}$ with $U \cap V \neq \varnothing$.

Every smooth atlas $\mathcal{A}$ of $\mathcal{M}$ is contained in a unique maximal smooth atlas ([Lee13, Proposition 1.17, p. 13]) – an atlas containing $\mathcal{A}$ which is not properly contained in any larger smooth atlas itself. A ***smooth structure*** on a topological manifold $\mathcal{M}$ is a maximal smooth atlas. A ***smooth manifold*** is a topological manifold endowed with a smooth structure. A coordinate chart of a smooth manifold is called a ***smooth coordinate chart***.

Let $\mathcal{M}^m$ and $\mathcal{N}^n$ be smooth manifolds. A map $F \colon \mathcal{M} \to \mathcal{N}$ is called a ***smooth map*** if for every $p \in \mathcal{M}$ there are smooth coordinate charts (U,φ) containing p and (V,ψ) containing $F(p)$ such that $F(U) \subset V$ and

$$\widehat{F} := \psi \circ F \circ \varphi^{-1} \in C^\infty(\varphi(U), \psi(V))$$

where $\varphi(U)$ is an open subset of $\mathbb{R}^m$ and $\psi(V)$ is an open subset of $\mathbb{R}^n$. The function $\widehat{F}$ is called a ***coordinate representation of F***. Note that this definition does not depend on the choice of coordinate charts because the transition maps are smooth. The set of smooth maps between manifolds will be denoted by $C^\infty(\mathcal{M},\mathcal{N})$. In the special case when $\mathcal{N} = \mathbb{R}$ we set

$$C^\infty(\mathcal{M}) := C^\infty(\mathcal{M}, \mathbb{R}).$$

A ***homeomorphism from $\mathcal{M}$ to $\mathcal{N}$*** is a continuous bijective map $F \colon \mathcal{M} \to \mathcal{N}$ that has a continuous inverse. A smooth bijective map $F \colon \mathcal{M} \to \mathcal{N}$ that has a smooth inverse is a ***diffeomorphism from $\mathcal{M}$ to $\mathcal{N}$***.

Now we will introduce a "local linear approximation" of a given smooth m-dimensional manifold $\mathcal{M}$. Let p be a point of $\mathcal{M}$. A linear map $v \colon C^\infty(\mathcal{M}) \to \mathbb{R}$ is called a ***derivation at p*** if it satisfies the following product rule

$$v(fg) = f(p)v(g) + g(p)v(f) \tag{1.3.22}$$

for all $f, g \in C^\infty(\mathcal{M})$. The set of all derivations at p is called the ***tangent space at p*** and is denoted by $T_p\mathcal{M}$. An element of $T_p\mathcal{M}$ is called a ***tangent vector at p***. It can be shown that $T_p\mathcal{M}$ is an m-dimensional real vector space. A basis of $T_p\mathcal{M}$ can be given through a smooth coordinate chart $(U, \varphi) = (U, (u^\alpha))$ containing p by defining the derivations $\frac{\partial}{\partial u^\alpha}\big|_p$ as

$$\frac{\partial}{\partial u^\alpha}\bigg|_p (f) := \frac{\partial}{\partial u^\alpha}\bigg|_{\varphi(p)} (f \circ \varphi^{-1})$$

for $f \in C^\infty(U)$ and $\alpha = 1, 2, \ldots, m$ (see [Lee13, Proposition 3.15, p. 61]). Note that on the right-hand side the symbol $\frac{\partial}{\partial u^\alpha}$ means the partial derivative with respect to the coordinate u^α of a real-valued function, whereas on the left-hand side it denotes a derivation. These basis tangent vectors are called ***coordinate vectors*** and form the ***coordinate basis for $T_p\mathcal{M}$***. Thus, any tangent vector $v \in T_p\mathcal{M}$ can be written uniquely as

$$v = v^\alpha \frac{\partial}{\partial u^\alpha}\bigg|_p$$

where $v^\alpha \in \mathbb{R}$ and the summation convention is used if an upper index "in the denominator" is understood as a lower index. That is, the right-hand side is understood to be summed over $\alpha = 1, 2, \ldots, m$.

The ***tangent bundle of $\mathcal{M}$*** is the disjoint union of all tangent spaces $T_p\mathcal{M}$ to $\mathcal{M}$, that is,

$$T\mathcal{M} := \bigsqcup_{p \in \mathcal{M}} T_p\mathcal{M}.$$

Here the ***disjoint union*** $\bigsqcup_{p\in\mathcal{M}} T_p\mathcal{M}$ is the set $\bigcup_{p\in\mathcal{M}} \{p\} \times T_p\mathcal{M}$. Thus, an element of $T\mathcal{M}$ is of the form (p, v) where $v \in T_p\mathcal{M}$. The tangent bundle of a smooth m-dimensional manifold is itself a smooth manifold of dimension $2m$. A local smooth coordinate chart $(U, \varphi) = (U, (u^\alpha))$ of $\mathcal{M}$ extends to a smooth coordinate chart of $T\mathcal{M}$ in the following way. Let $\pi\colon T\mathcal{M} \to \mathcal{M}$ be the natural projection map defined by $\pi(p, v) := p$. The set $\pi^{-1}(U) \subset T\mathcal{M}$ is the set of all tangent vectors of $\mathcal{M}$ at points of U. Define $\widetilde{\varphi}\colon \pi^{-1}(U) \to \mathbb{R}^{2m}$ by

$$\widetilde{\varphi}\left(p, v^\alpha \frac{\partial}{\partial u^\alpha}\bigg|_p\right) := \left(u^1(p), u^2(p), \ldots, u^m(p), v^1, v^2, \ldots, v^m\right). \tag{1.3.23}$$

Its image set is $\varphi(U_p) \times \mathbb{R}^m$. It can be shown that this map is a smooth coordinate chart and all such maps form a smooth atlas for $T\mathcal{M}$, turning it into a smooth manifold (see [Lee13, Proposition 3.18, p. 66]). The local coordinates given by (1.3.23) are called ***natural coordinates on*** $\boldsymbol{T\mathcal{M}}$.

A smooth map $F\colon \mathcal{M} \to \mathcal{N}$ between two smooth manifolds gives rise to a linear map between their respective tangent spaces. The ***differential of*** $\boldsymbol{F}$ ***at*** $\boldsymbol{p}$ is the linear map

$$dF_p\colon T_p\mathcal{M} \to T_{F(p)}\mathcal{N}$$

which for a fixed $v \in T_p\mathcal{M}$ is given by its action on $f \in C^\infty(\mathcal{M})$ by

$$dF_p(v)(f) := v(f \circ F).$$

If one considers the differential at each point of $\mathcal{M}$ this map extends to a map between the tangent bundles, called the ***global differential of*** $\boldsymbol{F}$

$$dF\colon T\mathcal{M} \to T\mathcal{N}, (p, v) \mapsto (F(p), dF_p(v)).$$

If $(U, \varphi) = (U, (u^\alpha))$ is a smooth coordinate chart of $\mathcal{M}$ and $(V, \psi) = (V, (x^i))$ a smooth coordinate chart of $\mathcal{N}$ then the differential of F at p can be locally expressed by

$$dF_p\left(v^\alpha \left.\frac{\partial}{\partial u^\alpha}\right|_p\right) = v^\alpha \frac{\partial \widehat{F}^i}{\partial u^\alpha}(\widehat{p}) \left.\frac{\partial}{\partial x^i}\right|_{F(p)} \tag{1.3.24}$$

where $\widehat{F} := \psi \circ F \circ \varphi^{-1}$ and $\widehat{p} = \varphi(p)$ are the coordinate representations of F and p (see [Lee13, Equation (3.10), p. 63]). Here latin indices are summed over $i = 1, 2, \ldots, n$ and greek indices are summed over $\alpha = 1, 2, \ldots, m$.

A ***smooth immersion*** is a smooth map $X\colon \mathcal{M} \to \mathcal{N}$ whose differential $dX_p\colon T_p\mathcal{M} \to T_{X(p)}\mathcal{N}$ is injective at each point p in $\mathcal{M}$. Equivalently, a smooth map $X\colon \mathcal{M} \to \mathcal{N}$ is a smooth immersion if and only if $\dim(\operatorname{im}(dX_p)) = m$ for all $p \in \mathcal{M}$. A ***smooth embedding of*** $\boldsymbol{\mathcal{M}}$ ***into*** $\boldsymbol{\mathcal{N}}$ is a smooth immersion $X\colon \mathcal{M} \to \mathcal{N}$ which also is a homeomorphism onto its image $X(\mathcal{M})$.

Suppose $X\colon \mathcal{M}^m \to \mathcal{N}^n$ is a smooth immersion. One can show that locally X is an embedding, that is, for each point $p \in \mathcal{M}$ there is a neighbourhood U of p in $\mathcal{M}$ such that $X\big|_U$ is a smooth embedding (see [Lee13, Theorem 4.25, p. 87]). As an open subset of $\mathcal{M}$ it is easy to show that U is a smooth m-dimensional manifold (by restricting the coordinate charts to U, see [Lee13, Example 1.26, p. 19]). Then the map $X\big|_U : U \to \mathcal{N}$ is a smooth embedding of a smooth m-dimensional manifold into a smooth n-dimensional manifold. Another result ([Lee13, Proposition 5.2, p. 99]) shows that the image of a smooth embedding is again a smooth manifold of the same dimension as the manifold in the domain of the embedding. Thus, the image $S := X\big|_U(U) \subset \mathcal{N}$ is a smooth m-manifold in the ambient smooth n-manifold $\mathcal{N}$ wherefore it makes sense to consider the tangent space $T_{X(p)}S$ to S at a point $X(p) \in \mathcal{N}$.

Lemma 1.3.1
Let $X\colon \mathcal{M}^m \to \mathcal{N}^n$ be a smooth immersion, p a point in $\mathcal{M}$ and $U \subset \mathcal{M}$ an open neighbourhood of p. Then the tangent space to $S := X(U)$ at $X(p)$ is given by

$$T_{X(p)}S = d\left(X\big|_U\right)_p(T_pU).$$

Proof: By assumption $d\left(X\big|_U\right)_p : T_pU \to T_{X(p)}\mathcal{N}$ is an injective linear map and therefore bijective onto its image $d\left(X\big|_U\right)_p(T_pU)$. Hence, both vector spaces $T_{X(p)}S$ and $d\left(X\big|_U\right)_p(T_pU)$ are m-dimensional. For dimensional reasons we need only show that $d\left(X\big|_U\right)_p(T_pU) \subset T_{X(p)}S$ to finish the proof. Thus, suppose $w \in d\left(X\big|_U\right)_p(T_pU)$. Then there is $v \in T_pU$ such that $w = d\left(X\big|_U\right)_p(v)$. As a composition of linear maps, w is a linear map from $C^\infty(S)$ to $\mathbb{R}$. We need to show the product rule (1.3.22). Let $f, g \in C^\infty(S)$. Then $f \circ X\big|_U, g \circ X\big|_U \in C^\infty(U)$. Unraveling the definitions and using (1.3.22) for $v \in T_pU$ we get

$$\begin{aligned} w(fg) &= d\left(X\big|_U\right)_p(v)(fg) \\ &= v((fg) \circ X\big|_U) \\ &= v((f \circ X\big|_U)(g \circ X\big|_U)) \\ &= (f \circ X\big|_U)(p)v(g \circ X\big|_U) + (g \circ X\big|_U)(p)v(f \circ X\big|_U) \\ &= f(X(p))d\left(X\big|_U\right)_p(v)(g) + g(X(p))d\left(X\big|_U\right)_p(v)(f) \\ &= f(X(p))w(g) + g(X(p))w(f). \end{aligned}$$

Therefore, w is a derivation at $X(p)$ and so $w \in T_{X(p)}S$. □

Furthermore, the injectivity of the differential of a smooth immersion immediately yields the next result.

Corollary 1.3.2
Let $X\colon \mathcal{M}^m \to \mathcal{N}^n$ be a smooth immersion and $(U, (u^\alpha))$ a smooth coordinate chart of $\mathcal{M}$ at some

point p. Then

$$\left(dX_p\left(\frac{\partial}{\partial u^1}\bigg|_p\right), dX_p\left(\frac{\partial}{\partial u^2}\bigg|_p\right), \ldots, dX_p\left(\frac{\partial}{\partial u^m}\bigg|_p\right)\right)$$

is an ordered basis for the m-dimensional subspace $T_{X(p)}X(U)$ of the n-dimensional tangent space $T_{X(p)}\mathcal{N}$ to the ambient manifold $\mathcal{N}$ at $X(p)$.

1.3.2 Finsler manifolds

Let $\mathcal{N}$ be a smooth n-dimensional manifold and $T\mathcal{N} = \bigsqcup_{p\in\mathcal{N}} T_p\mathcal{N}$ be its tangent bundle. The subset $o := \{(p,0) \in T\mathcal{N}\}$ of $T\mathcal{N}$ is called the ***zero section*** of $T\mathcal{N}$.

A non-negative function $F\colon T\mathcal{N} \to [0,\infty)$ is called a ***Finsler metric*** if the following three conditions are satisfied:

- ***Regularity*** : $F \in C^\infty(T\mathcal{N}\setminus o) \cap C^0(T\mathcal{N})$.
- ***Positive homogeneity***: $F(p,tv) = tF(p,v)$ for all $t > 0$ and $(p,v) \in T\mathcal{N}$.
- ***Ellipticity***: The coefficients $g^F_{ij}(p,v) := \left(\frac{1}{2}F^2\right)_{y^iy^j}(p,v)$ form a positive definite matrix for any $(p,v) \in T\mathcal{N}\setminus o$.

The third condition is to be understood in the following way. Recall that a smooth coordinate chart $(U,(x^1,x^2,\ldots,x^n))$ containing $p \in \mathcal{N}$ induces natural coordinates $\widetilde{\varphi} = (x^i,y^i)$ on the tangent bundle by $v = y^i \frac{\partial}{\partial x^i}\big|_p \in T_p\mathcal{N}$ (see (1.3.23)). Then in local coordinate representation F is given by

$$F(p,v) = \widehat{F}(x^1,x^2,\ldots,x^n,y^1,y^2,\ldots,y^n).$$

It is with respect to the local expression $\widehat{F} = F\circ\widetilde{\varphi}^{-1}$ that we take the derivatives with respect to y^i and y^j. The collection of coefficients $\left(g^F_{ij}(p,v)\right)_{ij}$ are called the ***fundamental tensor*** and locally define the ***fundamental form*** by

$$g^F\big|_{(p,v)}(u,w) := \frac{1}{2}\frac{\partial^2}{\partial s\partial t}\bigg|_{s=t=0} F^2(p,v+su+tw) = g^F_{ij}(p,v)u^iw^j$$

where $u = u^i\frac{\partial}{\partial x^i}\big|_p$, $w = w^i\frac{\partial}{\partial x^i}\big|_p \in T_p\mathcal{N}$. A Finsler metric is called ***reversible*** if $F(p,v) = F(p,-v)$ for all $(p,v) \in T\mathcal{N}$.

A smooth n-manifold $\mathcal{N}$ together with a Finsler metric F is called a ***Finsler manifold*** and will be denoted by $(\mathcal{N},F)$ or $(\mathcal{N}^n,F)$.

Let $p \in \mathcal{N}$ be fixed. One can show that the three conditions for a Finsler metric imply that $F(p, \cdot)$ is an asymmetric (only positive homogeneous) norm on the tangent space $T_p\mathcal{N}$, that is, the following additional two conditions hold (see [BCS00, Theorem 1.2.2, pp. 7–9]).

- ***Positivity***: $F(p, v) > 0$ whenever $v \neq 0 \in T_p\mathcal{N}$.
- ***Triangle inequality***: $F(p, v_1 + v_2) \leq F(p, v_1) + F(p, v_2)$ for all $v_1, v_2 \in T_p\mathcal{N}$ where equality holds if and only if $v_2 = \alpha v_1$ for some $\alpha \geq 0$.

If, in addition, the Finsler metric is reversible then one has absolute homogeneity and the tangent space $T_p\mathcal{N}$ becomes a normed space with norm $F(p, \cdot)$ in the usual sense of functional analysis. The positive homogeneity and the triangle inequality imply that $F(\cdot, \cdot)$ is a convex function in its second component.

A Finsler metric F is called a ***local Minkowski metric*** if for each $p \in \mathcal{N}$ there are smooth coordinate charts $(U_p, (x^1, x^2, \ldots, x^n))$ such that the coordinate representation of F at p is independent of x^i, that is,

$$F(p, v) = \widehat{F}(x^1, x^2, \ldots, x^n, y^1, y^2, \ldots, y^n) = \widehat{F}(y^1, y^2, \ldots, y^n)$$

where $v = y^i \left.\frac{\partial}{\partial x^i}\right|_p \in T_p\mathcal{N}$. A Finsler manifold $(\mathcal{N}, F)$ where F is a local Minkowski metric is called ***locally Minkowskian***.

Chapter 2

The Busemann–Hausdorff definition of area in Finsler geometry

In the first section of this chapter, we formalise the notion of Finsler volume introduced by Busemann in [Bus47]. Subsequently, the Finsler area of an immersion arises. Similarly to Overath [Ove13] we introduce the m-dimensional Busemann–Hausdorff area integrand which in turn leads to the m-dimensional Busemann–Hausdorff area density.

The second section mainly contains an analysis of Burago and Ivanov's paper *'Minimality of planes in normed spaces'* [BI12]. In their work they proved the convexity of the two-dimensional Busemann–Hausdorff area density. Using multilinear algebra we introduce the concept of a calibrator for an area density. We show that the convexity of a density is equivalent to the existence of such calibrators. Subsequently, the defining inequality for a calibrator will be reformulated into the language of convex geometry and an explicit construction for calibrators will be given. This finally results in an inequality for polytopes on the two-dimensional plane.

Note that in contrast to Euclidean space or more generally Riemannian manifolds, there is no unique notion of volume when dealing merely with a normed space or more generally a Finsler manifold. The Busemann–Hausdorff definition is only one of many possible choices of volume in the setting of a Finsler manifold. An alternative is the Holmes–Thompson volume ([Tho96, Chapter 6]) which we do not address here.

2.1 The m-dimensional Busemann–Hausdorff area density

Suppose $(\mathcal{M}^m, F)$ is a Finsler manifold of dimension m. Let $p \in \mathcal{M}$ be an arbitrary but fixed point and (U_p, φ) be a smooth coordinate chart containing p where we denote the coordinate functions of φ

as $(u^1, u^2, \ldots, u^m)$. Then we define the ***(closed) Finslerian unit ball at p*** as

$$B_p^F := \{v \in T_p\mathcal{M} \mid F(p,v) \leq 1\} \subset T_p\mathcal{M}.$$

The Busemann–Hausdorff volume form is given by the volume ratio of the Euclidean and the Finslerian unit ball. To make sense of the latter notion we need to use the local coordinate chart (U_p, φ) and the Hausdorff measure in $\mathbb{R}^m$. Let $\pi\colon T\mathcal{M} \to \mathcal{M}, (q,v) \to q$ be the natural projection map of the tangent bundle and recall that φ induces natural coordinates $(\pi^{-1}(U_p), \widetilde{\varphi})$ on the tangent bundle $T\mathcal{M}$ by (1.3.23), that is,

$$\widetilde{\varphi}\colon \pi^{-1}(U_p) \to \varphi(U_p) \times \mathbb{R}^m, \left(q, v^\alpha \frac{\partial}{\partial u^\alpha}\bigg|_q\right) \mapsto \left(u^1(q), u^2(q), \ldots, u^m(q), v^1, v^2, \ldots, v^m\right).$$

The Finslerian unit ball at p in the tangent bundle – the set $\{(p,v) \mid v \in B_p^F\} \subset T\mathcal{M}$ – is contained in $\pi^{-1}(U_p)$. Let $\pi_2 : \mathbb{R}^m \times \mathbb{R}^m \to \mathbb{R}^m, (x,y) \mapsto y$ be the projection map onto the second component. Then the Finslerian unit ball at p corresponds to the set

$$\widetilde{B}_p^F := \pi_2 \circ \widetilde{\varphi}\left(\{(p,v) \mid v \in B_p^F\}\right) \subset \mathbb{R}^m.$$

Note that by using the natural coordinates one can rewrite this set as

$$\widetilde{B}_p^F = \left\{ v = (v^1, v^2, \ldots, v^m) \in \mathbb{R}^m \;\middle|\; F\left(p, v^\alpha \frac{\partial}{\partial u^\alpha}\bigg|_p\right) \leq 1 \right\}$$

The ***m-dimensional Busemann–Hausdorff volume form*** is given by

$$dV_F(p) := \sigma_F(p) du^1 \wedge du^2 \wedge \cdots \wedge du^m \tag{2.1.1}$$

where

$$\sigma_F(p) := \frac{\varepsilon_m}{\mathcal{H}^m\left(\widetilde{B}_p^F\right)}. \tag{2.1.2}$$

Here $\varepsilon_m := \mathcal{H}^m(B_1^m(0))$ denotes the volume of the m-dimensional Euclidean unit ball in $\mathbb{R}^m$ and $\mathcal{H}^m$ is the m-dimensional Hausdorff measure on $\mathbb{R}^m$.

The ***Busemann–Hausdorff volume*** of a subset $\Omega \subset \mathcal{M}$ is defined as

$$\mathrm{vol}_F(\Omega) := \int_{p\in\Omega} dV_F(p). \tag{2.1.3}$$

If F is a local Minkowski metric, that is, independent of the point p and $\mathcal{M} = \mathbb{R}^m$ then the Busemann–Hausdorff volume of the Finslerian unit ball $B^F = \{v \in \mathbb{R}^m \mid F(v) \leq 1\}$ equals the volume of the Euclidean unit ball

$$\begin{aligned}
\mathrm{vol}_F(B^F) &= \int_{p\in\Omega} dV_F(p)\\
&= \int_{p\in B^F} \frac{\varepsilon_m}{\mathcal{H}^m(\widetilde{B}^F_p)} du^1 \wedge du^2 \wedge \cdots \wedge du^m\\
&= \int_{p\in B^F} \frac{\varepsilon_m}{\mathcal{H}^m(\widetilde{B}^F)} du^1 \wedge du^2 \wedge \cdots \wedge du^m\\
&= \frac{\varepsilon_m}{\mathcal{H}^m(B^F)} \int_{p\in B^F} du^1 du^2 \cdots du^m\\
&= \frac{\varepsilon_m}{\mathcal{H}^m(B^F)} \mathcal{H}^m(B^F)\\
&= \varepsilon_m.
\end{aligned}$$

It is possible to show that for absolutely homogeneous Finsler metrics F, the Finsler manifold $(\mathcal{M}, F)$ inherits a metric space structure ([BCS00, §6.2, p. 145]). Busemann showed in [Bus47] that the Finsler volume vol_F of a set $\Omega \subset \mathcal{M}$ coincides with the Hausdorff measure induced by this metric space structure. Using the Busemann–Hausdorff definition of volume we can introduce a way of measuring surface area of a submanifold immersed into an ambient manifold.

Let $m \leq n$ and $X\colon \mathcal{M}^m \to \mathcal{N}^n$ be a smooth immersion of a smooth m-manifold $\mathcal{M}$ into a Finsler n-manifold $(\mathcal{N}, F)$. Then $\mathcal{M}$ inherits a ***pull-back Finsler metric*** through

$$X^*F(p, v) := F(X(p), dX_p(v)) \tag{2.1.4}$$

for $(p, v) \in T_p\mathcal{M}$. The map $X^*F\colon T\mathcal{M} \to [0, \infty)$ is indeed a Finsler metric, because X is smooth and its differential at each $p \in \mathcal{M}$ injective. The ***Busemann–Hausdorff area of the immersion X*** for a measurable subset $\Omega \subset \mathcal{M}$ is given by

$$\mathrm{area}^F_\Omega(X) := \int_{p\in\Omega} dV_{X^*F}(p). \tag{2.1.5}$$

Therein, $dV_{X^*F}(p)$ is given by the formula (2.1.1).

Let $p \in \mathcal{M}$ be an arbitrary but fixed point and $(U_p, (u^1, u^2, \dots, u^m))$ be a smooth coordinate chart containing p. The local Busemann–Hausdorff integrand $\sigma_{X^*F}(p)$ can be rewritten as

$$\sigma_{X^*F}(p) = \frac{\varepsilon_m}{\mathcal{H}^m\left(\left\{v = (v^1, v^2, \dots, v^m) \in \mathbb{R}^m \;\middle|\; X^*F\left(p, v^\alpha \frac{\partial}{\partial u^\alpha}\big|_p\right) \leq 1\right\}\right)}$$

$$= \frac{\varepsilon_m}{\mathcal{H}^m\left(\left\{v \in \mathbb{R}^m \,\middle|\, F\left(X(p), v^\alpha dX_p\left(\frac{\partial}{\partial u^\alpha}\Big|_p\right)\right) \leq 1\right\}\right)}$$

Therefore, we define the map

$$\begin{aligned} a_m^F \colon & \bigsqcup_{q\in\mathcal{N}} GC_m(T_q\mathcal{N}) \to \mathbb{R}_+, \\ & (q, w_1 \wedge w_2 \wedge \cdots \wedge w_m) \mapsto \frac{\varepsilon_m}{\mathcal{H}^m\left(\{v = (v^1, v^2, \ldots, v^m) \in \mathbb{R}^m \mid F\left(q, v^\alpha w_\alpha\right) \leq 1\}\right)} \end{aligned} \tag{2.1.6}$$

Recall that $GC_m(T_q\mathcal{N}) \subset \bigwedge^m(T_q\mathcal{N})$ denotes the set of simple m-vectors on $T_q\mathcal{N}$. Note that this map is well-defined; for if $\widetilde{w}_1 \wedge \widetilde{w}_2 \wedge \cdots \wedge \widetilde{w}_m$ is another representation of $w_1 \wedge w_2 \wedge \cdots \wedge w_m$ then the injectivity of the Plücker embedding (Proposition 1.1.9) implies that the vectors $\{\widetilde{w}_i\}_{i=1}^m$ and $\{w_i\}_{i=1}^m$ span the same m-dimensional subspace. Therefore the denominator in (2.1.6) does not change value.

We call a_m^F the ***m-dimensional Busemann–Hausdorff area integrand*** or ***m-dimensional Finsler area integrand***. Notice the similarity to the Finsler area integrand defined by Overath in [Ove13, Equation (2.1.5), p. 68]. Therein, Overath uses the representation of the differential in local coordinates through its Jacobian matrix (see (1.3.24)) and defines the Finsler area integrand locally over $\mathcal{N} \times \mathbb{R}^{n\times m}$. In the present work we use the above definition to apply Burago and Ivanov's result. Note that our definition is not a local one – which contrasts [Ove13].

In local coordinates about p in $\mathcal{M}$ we thereby get

$$\sigma_{X^*F}(p) = a_m^F\left(X(p), dX_p\left(\frac{\partial}{\partial u^1}\bigg|_p\right) \wedge dX_p\left(\frac{\partial}{\partial u^2}\bigg|_p\right) \wedge \cdots \wedge dX_p\left(\frac{\partial}{\partial u^m}\bigg|_p\right)\right).$$

Recall from the previous chapter that locally the image of the immersion X is a submanifold of dimension m in the ambient n-manifold $\mathcal{N}$. By Corollary 1.3.2 we know that $dX_p\left(\frac{\partial}{\partial u^1}\big|_p\right) \wedge dX_p\left(\frac{\partial}{\partial u^2}\big|_p\right) \wedge \cdots \wedge dX_p\left(\frac{\partial}{\partial u^m}\big|_p\right)$ corresponds to the tangent space $T_{X(p)}X(U_p)$ through the Plücker embedding (Proposition 1.1.9). Thus, we may rewrite the expression (2.1.5) for the Busemann–Hausdorff area of an immersion X as

$$\operatorname{area}_\Omega^F(X) = \int_{p\in\Omega} a_m^F\left(X(p), dX_p\left(\frac{\partial}{\partial u^1}\bigg|_p\right) \wedge \cdots \wedge dX_p\left(\frac{\partial}{\partial u^m}\bigg|_p\right)\right) du^1 \wedge \cdots \wedge du^m. \tag{2.1.7}$$

Suppose now that $\mathcal{N} = \mathbb{R}^n$ and F is a reversible Finsler metric. Fix $q \in \mathbb{R}^n$. Then the tangent space $T_q\mathcal{N} = T_q\mathbb{R}^n$ is canonically isomorphic to $\mathbb{R}^n$ itself ([Lee13, Proposition 3.13, p. 59]). Since F is a reversible Finsler metric, the function $\|\cdot\|_V := F(q, \cdot)$ defines a norm on the vector space $V := \mathbb{R}^n \cong T_q\mathbb{R}^n$. Denote the closed unit ball by $B := B_q^F = \{v \in V \mid F(q, v) \leq 1\}$. Then $(V, \|\cdot\|_V)$ is an n-dimensional normed space. More generally, we can make the following definition.

Definition 2.1.1

An ***m*-dimensional density** on an n-dimensional normed space $(V, \|\cdot\|_V)$ is a continuous function $A\colon GC_m(V) \to \mathbb{R}_+$ which is absolutely homogeneous, that is $A(\lambda\sigma) = |\lambda|\, A(\sigma)$ for all $\lambda \in \mathbb{R}$ and $\sigma \in GC_m(V)$.

Definition 2.1.2

The ***m*-dimensional Busemann–Hausdorff area density** $A^{bh} = A^{bh}_{V,m}\colon GC_m(V) \to \mathbb{R}_+$ in a normed space $(V, \|\cdot\|)$ is defined by

$$A^{bh}(\sigma) := \frac{\varepsilon_m}{\mathcal{H}^m(L_\sigma^{-1}(B))} \tag{2.1.8}$$

wherein $\sigma = v_1 \wedge v_2 \wedge \ldots \wedge v_m$ and $L_\sigma\colon \mathbb{R}^m \to V$ is the linear map that takes the standard basis $(e_1, e_2, \ldots, e_m)$ of $\mathbb{R}^m$ to $(v_1, v_2, \ldots, v_m)$, that is, $L_\sigma e_\alpha = v_\alpha$ for $\alpha = 1, 2, \ldots, m$.

Notice that $v_1, v_2, \ldots, v_m \in V$ are linearly independent (because $\sigma \in GC_m(V)$) and the map L_σ is always injective (and thus, bijective onto its image). Notice further that the map L_σ is *not* well-defined as a function of σ. However, the map A^{bh} is well-defined as we will see in Theorem 2.1.3.

The reason for the introduction of the function A^{bh} becomes clear if we restrict our point of view again to the specific normed space $(\mathbb{R}^n, F(q,\cdot))$ with unit ball $B = B^F_q$. Then for fixed $q \in \mathbb{R}^n = \mathcal{N}$ we calculate the value of $A^{bh}_{q,m} = A^{bh}$ to

$$\begin{aligned} A^{bh}_{q,m}(v_1 \wedge v_2 \wedge \ldots \wedge v_m) &= \frac{\varepsilon_m}{\mathcal{H}^m\left(\{w \in \mathbb{R}^m \mid Lw \in B\}\right)} \\ &= \frac{\varepsilon_m}{\mathcal{H}^m\left(\{w = w^\alpha e_\alpha \in \mathbb{R}^m \mid w^\alpha v_\alpha \in B^F_q\}\right)} \\ &= \frac{\varepsilon_m}{\mathcal{H}^m\left(\{w = w^\alpha e_\alpha \in \mathbb{R}^m \mid F(q, w^\alpha v_\alpha) \le 1\}\right)}. \end{aligned}$$

So, the m-dimensional Finsler area integrand given by (2.1.6) corresponds to the Busemann–Hausdorff area density through

$$a^F_m(q, v_1 \wedge v_2 \wedge \cdots \wedge v_m) = A^{bh}_{q,m}(v_1 \wedge v_2 \wedge \cdots \wedge v_m). \tag{2.1.9}$$

Theorem 2.1.3

The Busemann–Hausdorff area density A^{bh} is an m-dimensional density as defined in Definition 2.1.1.

Proof: First, we show that the map A^{bh} is both well-defined and absolutely homogeneous. Let

$\sigma = v_1 \wedge v_2 \wedge \cdots \wedge v_m$ and $\tau = w_1 \wedge w_2 \wedge \cdots \wedge w_m$ be two simple m-vectors and let there be a non-zero scalar λ such that $\sigma = \lambda\tau$. This means σ and τ span the same one-dimensional subspace of the mth exterior power. We recall that the Plücker embedding maps an m-dimensional subspace of V to the one-dimensional line spanned by the m-wedge of a basis of this subspace. In fact, because the Plücker embedding is injective (Proposition 1.1.9) we know that

$$\operatorname{span}\{v_1, v_2, \dots, v_m\} = P = \operatorname{span}\{w_1, w_2, \dots, w_m\}.$$

Then there is a bijective, linear map $T\colon P \to P$ which sends w_α to v_α. By virtue of Lemma 1.1.7 (v) we know, that $\sigma = \det T\tau$ and thus $\det T = \lambda$. Consider the two linear maps from the definition of the Busemann–Hausdorff density

$$\begin{aligned} L_\sigma &\colon \mathbb{R}^m \to P, e_\alpha \mapsto v_\alpha, \\ L_\tau &\colon \mathbb{R}^m \to P, e_\alpha \mapsto w_\alpha. \end{aligned}$$

Each map is bijective onto its image, so we can define $\widetilde{v}_\alpha := L_\tau^{-1}(v_\alpha) \in \mathbb{R}^m$. Then $\{\widetilde{v}_1, \widetilde{v}_2, \dots, \widetilde{v}_m\}$ is a basis of $\mathbb{R}^m$. Now consider the linear map $\widetilde{T}\colon \mathbb{R}^m \to \mathbb{R}^m$ sending the standard basis vector e_α to $\widetilde{v}_\alpha$. We can calculate

$$\begin{aligned} \widetilde{T}e_\alpha &= \widetilde{v}_\alpha = L_\tau^{-1}(v_\alpha) = L_\tau^{-1} \circ L_\sigma e_\alpha \\ \text{and} \quad L_\sigma e_\alpha &= v_\alpha = Tw_\alpha = T \circ L_\tau e_\alpha. \end{aligned}$$

Since this holds for $\alpha = 1, 2, \dots, m$, we get

$$L_\sigma = L_\tau \circ \widetilde{T} = T \circ L_\tau. \tag{2.1.10}$$

From (2.1.10) we conclude that $\det L_\tau \det \widetilde{T} = \det(L_\tau \circ \widetilde{T}) = \det(T \circ L_\tau) = \det T \det L_\tau$ and thus because L_τ is bijective we know

$$\det \widetilde{T} = \det T = \lambda.$$

Therefore, it holds that

$$\begin{aligned} \mathcal{H}^m\left(L_\sigma^{-1}(B)\right) &= \mathcal{H}^m\left(\left(L_\tau \circ \widetilde{T}\right)^{-1}(B)\right) = \mathcal{H}^m\left(\widetilde{T}^{-1} \circ L_\tau^{-1}(B)\right) \\ &= \left|\det \widetilde{T}^{-1}\right| \mathcal{H}^m\left(L_\tau^{-1}(B)\right) = \left|\lambda^{-1}\right| \mathcal{H}^m\left(L_\tau^{-1}(B)\right) \end{aligned} \tag{2.1.11}$$

where we used the fact that linear transformations change the m-dimensional Lebesgue measure of m-dimensional sets by a constant factor – the absolute value of the determinant of the transformation (see e.g. [For09, §5, Satz 2, p. 48]). By definition of the Busemann–Hausdorff area density we finally conclude with (2.1.11) that

$$A^{bh}(\lambda\tau) = A^{bh}(\sigma) = \frac{\varepsilon_m}{\mathcal{H}^m(L_\sigma^{-1}(B))} = |\lambda| \frac{\varepsilon_m}{\mathcal{H}^m(L_\tau^{-1}(B))} = |\lambda| A^{bh}(\tau) \tag{2.1.12}$$

Thus, we have both shown absolute homogeneity and well-definedness. For if $\lambda = 1$, that is, if $v_1 \wedge v_2 \wedge \cdots \wedge v_m = w_1 \wedge w_2 \wedge \cdots \wedge w_m$ are two representations of the same simple m-vector σ, then $A^{bh}(v_1 \wedge v_2 \wedge \cdots \wedge v_m) = A^{bh}(w_1 \wedge w_2 \wedge \cdots \wedge w_m)$ by (2.1.12).

For the continuity of the Busemann–Hausdorff area density we sketch a proof here and refer to [BI12] or [ÁPT04, Exercise 3.5, p. 11 and pp. 18-19]. We recall from Lemma 1.1.10 that the line spanned by a simple m-vector $\sigma \in GC_m(V)$ is in the image of the Plücker embedding. Then the injectivity of the latter mapping (Proposition 1.1.9) shows the existence of a unique m-dimensional plane $P_\sigma \in G_m(V)$ such that $\rho(P_\sigma) = [\sigma]_\sim$. Then $\operatorname{im}(L_\sigma) = P_\sigma$, so that

$$\mathcal{H}^m(L_\sigma^{-1}(B)) = \mathcal{H}^m(L_\sigma^{-1}(B \cap P_\sigma)).$$

The isomorphism $L_\sigma : \mathbb{R}^m \to P_\sigma$ induces an inner product on P_σ (see the introduction to Section 1.1.4 for details why this defines an inner product). Thus, we can choose an orthogonal basis of P_σ and write $\sigma = v_1 \wedge v_2 \wedge \cdots \wedge v_m$ where

$$\langle v_i, v_j \rangle_{P_\sigma} = 0 \quad \text{for } i \neq j.$$

The collection of convex bodies in $\mathbb{R}^m$, denoted by $\mathcal{C}_b(\mathbb{R}^m)$, turns into a metric space by means of the Hausdorff metric $d_\mathcal{H} = d_{\mathcal{H},\mathbb{R}^m}$ given by

$$d_\mathcal{H}(A, B) = \max\left(\sup_{a\in A} \inf_{b\in B} \|a - b\|_{R^m}, \sup_{b\in B} \inf_{a\in A} \|a - b\|_{R^m} \right)$$

where $\|\cdot\|_{\mathbb{R}^m}$ is the standard Euclidean norm on $\mathbb{R}^m$ (see [Tho96, Proposition 2.4.2, p. 61]). A result from convex geometry states that the m-dimensional Lebesgue measure $\mathcal{L}^m$ on $\mathbb{R}^m$ is continuous on the collection of compact convex sets in $\mathbb{R}^m$ with respect to the Hausdorff metric (see [Val68, Satz 12.7, p. 153] or [Bee74, p. 64]). Note that the m-dimensional Hausdorff measure equals the m-dimensional Lebesgue measure on $\mathbb{R}^m$ (see [EG92, Theorem 2, p. 70]).

Due to Lemma 1.2.16 (ii) and (iv), the set $L_\sigma^{-1}(B \cap P_\sigma)$ is a symmetric convex body in $\mathbb{R}^m$. Thus,

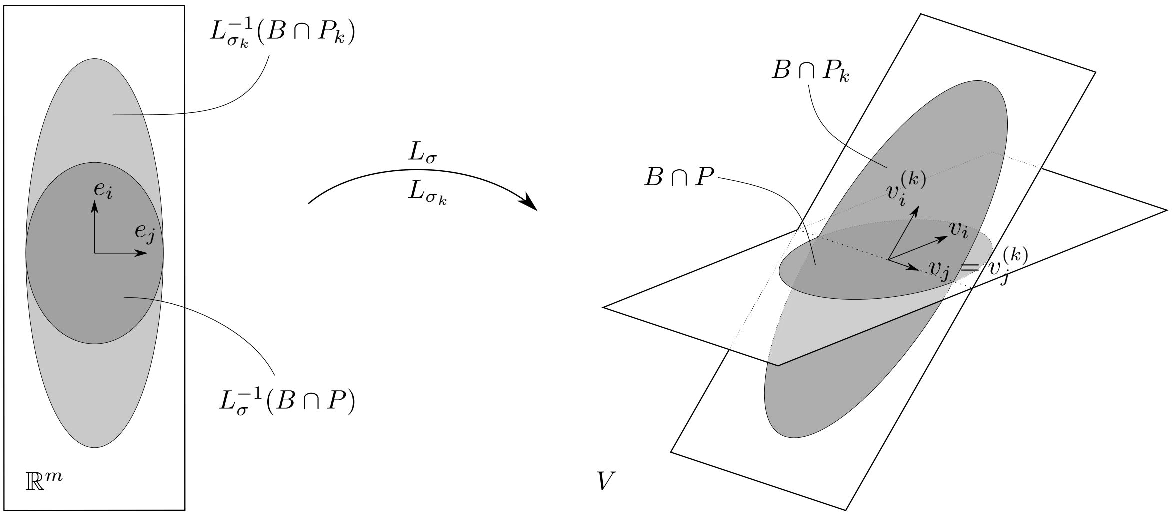


Figure 2.1: Planes corresponding to simple m-vectors where B is an ellipsoid

it suffices to prove the continuity of the map

$$\varphi\colon GC_m(V) \to \mathcal{C}_b(\mathbb{R}^m), \sigma = v_1 \wedge v_2 \wedge \cdots \wedge v_m \mapsto L_\sigma^{-1}(B \cap P_\sigma)$$

where we chose an orthogonal basis $\{v_1, v_2, \ldots, v_m\}$ of P_σ as stated above. Since P_σ and so $\langle\cdot,\cdot\rangle_{P_\sigma}$ is unique, this map is well-defined.

To show the continuity of φ, let $(\sigma_k)_{k\in\mathbb{N}} \subset GC_m(V)$ be a sequence converging to $\sigma \in GC_m(V)$. Denote the corresponding m-dimensional planes by P_k and P and write $\sigma_k = v_1^{(k)} \wedge v_2^{(k)} \wedge \cdots \wedge v_m^{(k)}$ and $\sigma = v_1 \wedge v_2 \wedge \cdots \wedge v_m$. Remember that we chose the bases of P_k and P to be orthogonal (in P_k and P respectively). Further, we can rotate them within the planes P_k and P, so that

$$\begin{aligned} \langle v_i, v_j\rangle_P = \left\langle v_i^{(k)}, v_j^{(k)}\right\rangle_{P_k} &= 0 \quad \text{for } i \neq j \text{ and all } k \in \mathbb{N} \\ \text{and} \quad v_i^{(k)} &\xrightarrow[k\to\infty]{} v_i \quad \text{for } i = 1, 2, \ldots, m. \end{aligned}$$

Here the inner products $\langle\cdot,\cdot\rangle_P$ and $\langle\cdot,\cdot\rangle_{P_k}$ are induced by the isomorphisms $L_\sigma\colon \mathbb{R}^m \to P$ and $L_{\sigma_k}\colon \mathbb{R}^m \to P_k$ respectively. We recall that the Plücker embedding is a topological embedding, that is, a homeomorphism onto its image (see remark after Proposition 1.1.9). Therefore, if $\sigma_k \xrightarrow[k\to\infty]{} \sigma$ then $P_k \xrightarrow[k\to\infty]{} P$ in $G_m(V)$.

We need to estimate

$$
\begin{aligned}
d_{\mathcal{H}}\left(L_{\sigma_k}^{-1}(B)\,,\,L_{\sigma}^{-1}(B)\right) &= \max\left(\sup_{a\in L_{\sigma_k}^{-1}(B)}\ \inf_{b\in L_{\sigma}^{-1}(B)}\|a-b\|_{R^m},\ \sup_{b\in L_{\sigma}^{-1}(B)}\ \inf_{a\in L_{\sigma_k}^{-1}(B)}\|a-b\|_{R^m}\right)\\
&= \max\left(\sup_{x\in B\cap P_k}\ \inf_{y\in B\cap P}\left\|L_{\sigma_k}^{-1}x-L_{\sigma}^{-1}y\right\|_{R^m},\ \sup_{y\in B\cap P}\ \inf_{x\in B\cap P_k}\left\|L_{\sigma_k}^{-1}x-L_{\sigma}^{-1}y\right\|_{R^m}\right).
\end{aligned}
$$

Hence, let $x\in B\cap P_k$ und $y\in B\cap P$. Further, let $\Pi_k : V\to P_k$ be the projection onto P_k. Then

$$
\begin{aligned}
\left\|L_{\sigma_k}^{-1}x-L_{\sigma}^{-1}y\right\|_{\mathbb{R}^m} &\le \left\|L_{\sigma_k}^{-1}x-L_{\sigma_k}^{-1}\Pi_k y\right\|_{\mathbb{R}^m}+\left\|L_{\sigma_k}^{-1}\Pi_k y-L_{\sigma}^{-1}y\right\|_{\mathbb{R}^m}\\
&= \left\|L_{\sigma_k}^{-1}\Pi_k(x-y)\right\|_{\mathbb{R}^m}+\left\|\left(L_{\sigma_k}^{-1}\Pi_k-L_{\sigma}^{-1}\right)y\right\|_{\mathbb{R}^m}\\
&\le \left\|L_{\sigma_k}^{-1}\Pi_k\right\|_{\mathcal{L}(V,\mathbb{R}^m)}\|x-y\|_{\mathbb{R}^m}+\left\|\left(L_{\sigma_k}^{-1}\Pi_k-L_{\sigma}^{-1}\right)y\right\|_{\mathbb{R}^m}
\end{aligned}
$$

and similarly,

$$
\left\|L_{\sigma_k}^{-1}x-L_{\sigma}^{-1}y\right\|_{\mathbb{R}^m}\le\left\|L_{\sigma}^{-1}\Pi\right\|_{\mathcal{L}(V,\mathbb{R}^m)}\|x-y\|_{\mathbb{R}^m}+\left\|\left(L_{\sigma_k}^{-1}-L_{\sigma}^{-1}\Pi\right)x\right\|_{\mathbb{R}^m}
$$

where $\Pi : V\to P$ is the projection onto P. Thus, we can estimate

$$
\begin{aligned}
\inf_{y\in B\cap P}\left\|L_{\sigma_k}^{-1}x-L_{\sigma}^{-1}y\right\|_{\mathbb{R}^m} &\le \inf_{y\in B\cap P}\left(\left\|L_{\sigma_k}^{-1}\Pi_k\right\|_{\mathcal{L}(V,\mathbb{R}^m)}\|x-y\|_{\mathbb{R}^m}+\left\|\left(L_{\sigma_k}^{-1}\Pi_k-L_{\sigma}^{-1}\right)y\right\|_{\mathbb{R}^m}\right)\\
&\le \inf_{y\in B\cap P}\left\|L_{\sigma_k}^{-1}\Pi_k\right\|_{\mathcal{L}(V,\mathbb{R}^m)}\|x-y\|_{\mathbb{R}^m}+\sup_{y\in B\cap P}\left\|\left(L_{\sigma_k}^{-1}\Pi_k-L_{\sigma}^{-1}\Pi\right)y\right\|_{\mathbb{R}^m}\\
&\le \left\|L_{\sigma_k}^{-1}\Pi_k\right\|_{\mathcal{L}(V,\mathbb{R}^m)}\operatorname{dist}\left(x,B\cap P\right)+\left\|L_{\sigma_k}^{-1}\Pi_k-L_{\sigma}^{-1}\Pi\right\|_{\mathcal{L}(V,\mathbb{R}^m)}
\end{aligned}
\tag{2.1.13}
$$

and similarly,

$$
\inf_{x\in B\cap P_k}\left\|L_{\sigma_k}^{-1}x-L_{\sigma}^{-1}y\right\|_{\mathbb{R}^m}\le\left\|L_{\sigma}^{-1}\Pi\right\|_{\mathcal{L}(V,\mathbb{R}^m)}\operatorname{dist}\left(y,B\cap P_k\right)+\left\|L_{\sigma_k}^{-1}\Pi_k-L_{\sigma}^{-1}\Pi\right\|_{\mathcal{L}(V,\mathbb{R}^m)}. \tag{2.1.14}
$$

Therefore, (2.1.13) and (2.1.14) together yield

$$
\begin{aligned}
&d_{\mathcal{H},\mathbb{R}^m}\left(L_{\sigma_k}^{-1}(B)\,,\,L_{\sigma}^{-1}(B)\right)\\
&\qquad\le \max\left(\left\|L_{\sigma_k}^{-1}\Pi_k\right\|_{\mathcal{L}(V,\mathbb{R}^m)},\left\|L_{\sigma}^{-1}\Pi\right\|_{\mathcal{L}(V,\mathbb{R}^m)}\right)d_{\mathcal{H},V}\left(B\cap P_k\,,\,B\cap P\right)\\
&\qquad\quad+\left\|L_{\sigma_k}^{-1}\Pi_k-L_{\sigma}^{-1}\Pi\right\|_{\mathcal{L}(V,\mathbb{R}^m)}.
\end{aligned}
\tag{2.1.15}
$$

Since we chose the bases $\{v_i^{(k)}\}_{i=1}^m \subset P_k$ and $\{v_i\}_{i=1}^m \subset P$ so that $v_i^{(k)} \xrightarrow[k\to\infty]{} v_i$, we know that

$$\begin{aligned}\|L_{\sigma_k} - L_\sigma\|_{\mathcal{L}(R^m,V)} &= \sup_{\sum_{i=1}^m (a^i)^2=1} \left\| \sum_{i=1}^m a^i \left(L_{\sigma_k} - L_\sigma\right) e_i \right\|_V \\ &\leq \sup_{\sum_{i=1}^m (a^i)^2=1} \sum_{i=1}^m |a^i| \left\| v_i^{(k)} - v_i \right\|_V \xrightarrow[k\to\infty]{} 0,\end{aligned}$$

that is, $L_{\sigma_k} \xrightarrow[k\to\infty]{} L_\sigma$ in $\mathcal{L}(\mathbb{R}^m, V)$. Note that the linear map $L_{\sigma_k}^{-1}\Pi_k : V \to \mathbb{R}^m$ is the Moore-Penrose pseudoinverse (or generalised inverse) of the linear map $L_{\sigma_k} : \mathbb{R}^m \to P_k \subset V$. All of the mappings L_{σ_k} are of rank m. Under these conditions, the map sending a matrix to its Moore-Penrose inverse is continuous (see [Rak91, Theorem 4.2, p. 166] and [Rak97]). Therefore, we have that $L_{\sigma_k}^{-1}\Pi_k \xrightarrow[k\to\infty]{} L_\sigma^{-1}\Pi$ in $\mathcal{L}(V, \mathbb{R}^m)$. Thus, the second summand in (2.1.15) vanishes for $k \to \infty$ and consequently, the factor $\max\left(\left\|L_{\sigma_k}^{-1}\Pi_k\right\|_{\mathcal{L}(V,\mathbb{R}^m)}, \left\|L_\sigma^{-1}\Pi\right\|_{\mathcal{L}(V,\mathbb{R}^m)}\right)$ is bounded above. It remains to show that

$$d_{\mathcal{H},V}\left(B \cap P_k\ , B \cap P\right) \xrightarrow[k\to\infty]{} 0.$$

This is true because $P_k \xrightarrow[k\to\infty]{} P$ in $G_m(V)$ as stated earlier (see Figure 2.1). So, φ and therewith, A^{bh} is continuous. $\square$

2.2 Convexity of the two-dimensional Busemann–Hausdorff area density

Definition 2.2.1

A density $A\colon GC_m(V) \to \mathbb{R}_+$ is called ***convex*** if it can be extended to an absolutely homogeneous, continuous and convex function on the entire vector space $\bigwedge^m V$.

Busemann proved in [Bus49] that the Busemann–Hausdorff area density is convex in codimension one, that is, $\dim(V) = m+1$, and the general case was left as a conjecture (see [Tho96, Problem 7.1.1, p. 310]). In their work [BI12] Burago and Ivanov proved the conjecture for the two-dimensional Busemann–Hausdorff area density. The next theorem is the main result of the present chapter. The rest of this section will be devoted to extensively discuss Burago and Ivanov's proof.

Theorem 2.2.2 (Convexity of the two-dimensional area density [BI12, Theorem 1, p. 630])

In every finite-dimensional normed space V, the two-dimensional Busemann–Hausdorff area density defined by (2.1.8) is convex on $\bigwedge^2 V$.

In what follows, we will present the geometric argument Burago and Ivanov used to prove their statement. Using the language of convex geometry the statement will be reformulated into an inequality for polytopes on the two-dimensional plane. In order to do so, let us first introduce the concept of a calibrator for an m-dimensional density.

Definition 2.2.3 ([BI12, Definition 2.1, p. 631])

Let V be a finite-dimensional vector space, $A\colon GC_m(V) \to \mathbb{R}_+$ an m-dimensional density and $P \subset V$ an m-dimensional subspace. A ***calibrator*** or ***calibrating form*** for P with respect to A is an exterior m-form $\omega \in \bigwedge^m(V^*) \cong (\bigwedge^m(V))^*$ such that for every simple m-vector $\sigma \in GC_m(V)$ one has $|\omega(\sigma)| \leq A(\sigma)$ and this inequality turns into equality if $\sigma \in GC_m(V) \cap \bigwedge^m(P)$.

Notice that $\bigwedge^m(P)$ is one-dimensional and therefore every m-vector in $\bigwedge^m(P)$ is simple. So that intersection $GC_m(V) \cap \bigwedge^m(P)$ really is just $\bigwedge^m(P)$ itself. The preceding notion of a calibrator is a powerful concept because the following central result holds.

Lemma 2.2.4

An m-dimensional density $A\colon GC_m(V) \to \mathbb{R}_+$ is convex on $\bigwedge^m(V)$ if and only if every m-dimensional plane $P \in G_m(V)$ admits a calibrator $\omega_P \in (\bigwedge^m(V))^*$ with respect to A.

Proof: Suppose the m-dimensional density A admits an absolutely homogeneous, continuous and convex extension $\mathcal{A}\colon \bigwedge^m(V) \to \mathbb{R}_+$. The following is due to [BRS12, p. 7]. The absolute homogeneity and convexity of the extension imply that $\mathcal{A}$ is a sublinear functional because for $\sigma, \tau \in \bigwedge^m(V)$ we have

$$\begin{aligned}\mathcal{A}(\sigma+\tau) &= \mathcal{A}\left(\frac{1}{2}(2\sigma)+\frac{1}{2}(2\tau)\right) \le \frac{1}{2}\mathcal{A}(2\sigma)+\frac{1}{2}\mathcal{A}(2\tau)\\ &= \mathcal{A}(\sigma)+\mathcal{A}(\tau).\end{aligned}$$

Let $P \in G_m(V)$ be arbitrary and spanned be a basis $\{v_1, v_2, \ldots, v_m\}$. Recall that the top exterior power $\bigwedge^m(P)$ is $\binom{m}{m} = 1$-dimensional. Note that any element of the top exterior power $\bigwedge^m(P)$ is of the form $\lambda v_1 \wedge v_2 \wedge \cdots \wedge v_m$, that is, any element is a simple m-vector or $\bigwedge^m(P) = GC_m(V) \cap \bigwedge^m(P)$. Let us define $\omega_{P,0}\colon \bigwedge^m(P) \to \mathbb{R}$ on this subspace by setting

$$\omega_{P,0}(\lambda v_1 \wedge v_2 \wedge \cdots \wedge v_m) := \lambda A(v_1 \wedge v_2 \wedge \cdots \wedge v_m).$$

This is a linear form on $\bigwedge^m(P)$ and for $\nu \in \bigwedge^m(P)$ we have

$$\begin{aligned}|\omega_{P,0}(\nu)| &= |\omega_{P,0}(\lambda v_1 \wedge v_2 \wedge \cdots \wedge v_m)|\\ &= |\lambda A(v_1 \wedge v_2 \wedge \cdots \wedge v_m)|\\ &= |\lambda| A(v_1 \wedge v_2 \wedge \cdots \wedge v_m)\\ &= A(\lambda v_1 \wedge v_2 \wedge \cdots \wedge v_m)\\ &= A(\nu) = \mathcal{A}(\nu)\end{aligned}$$

where we used the fact that the density A is non-negative and absolutely homogeneous. Therefore, $\omega_{P,0}$ is bounded above by $\mathcal{A}$ on the subspace $\bigwedge^m(P)$.

Then the Hahn-Banach theorem ([Tho96, Theorem 1.3.2, p. 33]) applied to $\omega_{P,0}$ and $\mathcal{A}$, asserts the existence of a linear form $w_P\colon \bigwedge^m(V) \to \mathbb{R}$ such that

$$\begin{aligned} &\omega_P(\sigma) = \omega_{P,0}(\sigma) &&\text{for all } \sigma \in \bigwedge^m(P)\\ \text{and}\quad &\omega_P(\sigma) \le \mathcal{A}(\sigma) &&\text{for all } \sigma \in \bigwedge^m(V).\end{aligned}$$

In particular, this implies

$$\begin{aligned} &|\omega_P(\sigma)| = A(\sigma) &&\text{for all } \sigma \in \bigwedge^m(P) = GC_m(V) \cap \bigwedge^m(P)\\ \text{and}\quad &|\omega_P(\sigma)| \le A(\sigma) &&\text{for all } \sigma \in GC_m(V).\end{aligned}$$

So this m-form is a calibrator for P with respect to A.

Conversely, suppose every m-dimensional plane P admits a calibrator $\omega_P \in (\bigwedge^m(V))^*$ with respect to A. Then the mapping $|\omega_P|$ is convex because ω_P is linear. In addition, for every $P \in G_m(V)$ the density A is an upper bound to $|\omega_P|$ on the domain $GC_m(V)$ of A, that is, it holds that

$$\begin{aligned} &|\omega_P(\sigma)| \leq A(\sigma) \quad \text{for all } \sigma \in GC_m(V) \\ \text{and} \quad &|\omega_P(\sigma)| = A(\sigma) \quad \text{for all } \sigma \in GC_m(V) \cap \textstyle\bigwedge^m(P). \end{aligned} \tag{2.2.16}$$

Construct the set of absolutely homogeneous, proper convex functions bounded above by A on $GC_m(V)$,

$$\mathcal{E} := \left\{ l\colon \textstyle\bigwedge^m(V) \to \mathbb{R}_+ \;\middle|\; l\big|_{GC_m(V)} \leq A, l \text{ is absolutely homogeneous and proper convex} \right\}.$$

Note that by the preceding argumentation $|\omega_P| \in \mathcal{E}$ for every $P \in G_m(V)$, thus $\mathcal{E}$ is non-empty. Then let us define the extension of A as its ***convex envelope***, that is, set

$$\mathcal{A}(\sigma) := \operatorname{conv}(A)(\sigma) := \sup_{l \in \mathcal{E}} \, l(\sigma)$$

for arbitrary $\sigma \in \bigwedge^m(V)$. The function $\mathcal{A}\colon \bigwedge^m(V) \to \mathbb{R}_+$ is convex because each $l \in \mathcal{E}$ is and

$$\begin{aligned} \mathcal{A}(t\sigma + (1-t)\nu) &= \sup_{l \in \mathcal{E}} \, l(t\sigma + (1-t)\nu) \leq \sup_{l \in \mathcal{E}} \, tl(\sigma) + (1-t)l(\nu) \\ &\leq t \sup_{l \in \mathcal{E}} \, l(\sigma) + (1-t) \sup_{l \in \mathcal{E}} \, l(\nu) \\ &= t\mathcal{A}(\sigma) + (1-t)\mathcal{A}(\nu). \end{aligned}$$

Further, $\mathcal{A}$ is absolutely homogeneous since for $\sigma \in \bigwedge^m(V), \lambda \in \mathbb{R}$

$$\mathcal{A}(\lambda\sigma) = \sup_{l \in \mathcal{E}} \, l(\lambda\sigma) = \sup_{l \in \mathcal{E}} \, |\lambda| l(\sigma) = |\lambda| \mathcal{A}(\sigma).$$

For an arbitrary simple m-vector $\sigma \in GC_m(V)$ there is a corresponding m-dimensional plane $Q \in G_m(V)$ such that $\sigma \in \bigwedge^m(Q)$, namely, if $\sigma = v_1 \wedge v_2 \wedge \cdots \wedge v_m$ then $Q := \operatorname{span}\{v_1, v_2, \ldots, v_m\}$. (Recall the Plücker embedding – in particular, Lemma 1.1.10.)

Thus, for the calibrator ω_Q of the plane Q the inequality

$$A(\sigma) = |\omega_Q(\sigma)| \leq \sup_{l \in \mathcal{E}} \, l(\sigma) = \mathcal{A}(\sigma)$$

holds. On the other hand, because each $l \in \mathcal{E}$ is bounded above by A on $GC_m(V)$, we have

$$\mathcal{A}(\sigma) = \sup_{l\in\mathcal{E}} l(\sigma) \le \sup_{l\in\mathcal{E}} A(\sigma) = A(\sigma).$$

by definition of $\mathcal{E}$. So, $\mathcal{A}\big|_{GC_m(V)} \equiv A$ and $\mathcal{A}$ is an absolutely homogeneous and convex extension of A. It remains to show the continuity of $\mathcal{A}$. Since each $l \in \mathcal{E}$ is proper the pointwise supremum over this set nowhere takes the value $-\infty$. On the set $GC_m(V)$ the function $\mathcal{A}$ is bounded above by A which in turn only takes finite values. Now let $\{e_1, e_2, \dots, e_n\}$ be a basis for V. Let $\sigma = \sum_{I\in\mathcal{I}} a_\sigma^I e_{i_1} \wedge e_{i_2} \wedge \cdots \wedge e_{i_m}$ be an arbitrary m-vector in $\bigwedge^m(V)$. As usual, $\mathcal{I}$ denotes the set of strictly increasing multi-indices of length m. Since $\mathcal{A}$ is convex and absolutely homogeneous, it is also sublinear by the same argument as in the first part of the proof. Thus, it follows that

$$\begin{aligned}\mathcal{A}(\sigma) &\le \sum_{I\in\mathcal{I}} \left|a_\sigma^I\right| \mathcal{A}(e_{i_1} \wedge e_{i_2} \wedge \cdots \wedge e_{i_m}) \\ &= \sum_{I\in\mathcal{I}} \left|a_\sigma^I\right| A(e_{i_1} \wedge e_{i_2} \wedge \cdots \wedge e_{i_m}) \le \widehat{C} \sum_{I\in\mathcal{I}} \left|a_\sigma^I\right|.\end{aligned}$$

where $\widehat{C} := \max_{I\in\mathcal{I}} A(e_{i_1} \wedge e_{i_2} \wedge \cdots \wedge e_{i_m}) < \infty$. Note that the Hölder-inequality for p-norms implies that

$$\sum_{I\in\mathcal{I}} \left|a_\sigma^I\right| = \sum_{I\in\mathcal{I}} \left|a_\sigma^I\right| \cdot 1 \le \left(\sum_{I\in\mathcal{I}} \left|a_\sigma^I\right|^2\right)^{\frac{1}{2}} \cdot \left(\sum_{I\in\mathcal{I}} 1\right)^{\frac{1}{2}} \le D \binom{n}{m}^{\frac{1}{2}} \|\sigma\|_{\bigwedge^m(V)}.$$

Therein, we used the equivalence of norms on the finite-dimensional vector space $\bigwedge^m(V)$, since $\left(\sum_{I\in\mathcal{I}} \left|a_\sigma^I\right|^2\right)^{\frac{1}{2}}$ is the Euclidean norm (see Corollary 1.1.12). Hence, we get for $C := \widehat{C}D$ that

$$\mathcal{A}(\sigma) \le C \binom{n}{m}^{\frac{1}{2}} \|\sigma\|_{\bigwedge^m(V)}. \tag{2.2.17}$$

Let $\varepsilon > 0$ be arbitrary and consider an m-vector $\tau \in \bigwedge^m(V)$ with $\|\sigma - \tau\|_{\bigwedge^m(V)} < \varepsilon$. Then using the sublinearity of $\mathcal{A}$ again yields

$$\begin{aligned}\mathcal{A}(\tau) &= \mathcal{A}((\sigma - \tau) + \sigma) \le \mathcal{A}(\sigma - \tau) + \mathcal{A}(\sigma) \\ &\le C \binom{n}{m}^{\frac{1}{2}} \left(\|\sigma - \tau\|_{\bigwedge^m(V)} + \|\sigma\|_{\bigwedge^m(V)}\right) \\ &< C \binom{n}{m}^{\frac{1}{2}} \left(\varepsilon + \|\sigma\|_{\bigwedge^m(V)}\right) =: \widetilde{C}(n, m, \varepsilon; \sigma) < \infty\end{aligned}$$

where we used (2.2.17) twice. Therefore, on a ball of radius $\varepsilon > 0$ centered at σ the convex function $\mathcal{A}$ is bounded above by a finite constant $\widetilde{C}(n, m, \varepsilon; \sigma)$. Then a result from convex analysis ([Cla90, Proposition 2.2.6, p. 34]) states that $\mathcal{A}$ is Lipschitz-continuous on this neighbourhood of σ. Since $\sigma \in \bigwedge^m(V)$ was arbitrary we find that, in particular, $\mathcal{A}$ is continuous on $\bigwedge^m(V)$. □

In what follows, we justify that we can make the subsequent simplifying assumption.

Assumption 2.2.5

The unit ball B of $(V, \|\cdot\|_V)$ is a symmetric polytope.

Denote the closed unit ball of the n-dimensional normed space $(V, \|\cdot\|_V)$ by B. Then the triangle inequality and absolute homogeneity of the norm imply that the unit ball B is convex and symmetric. It has non-empty interior because $0 \in B$ and $\|0\|_V = 0 < 1$. Since V is finite-dimensional B is a compact set (see [Tho96, Theorem 1.2.6, p. 30]). Therefore, B is a symmetric, convex, compact set with non-empty interior or in the terminology of Section 1.2.4 the unit ball B is a *symmetric* convex body.

As stated earlier, the collection of convex bodies in V, denoted by $\mathcal{C}_b = \mathcal{C}_b(V)$ and the Hausdorff metric $d_{\mathcal{H}} = d_{\mathcal{H},V}$ given by

$$d_{\mathcal{H}}(A, B) = \max\left(\sup_{a\in A}\inf_{b\in B}\|a-b\|_V, \sup_{b\in B}\inf_{a\in A}\|a-b\|_V\right)$$

turns into a metric space. A result from convex geometry (see [Tho96, Theorem 2.5.1, p. 64]) states that the set of polytopes is dense in this metric space. In particular, this means that for a convex body $C \in \mathcal{C}_b$ and any $\varepsilon > 0$ there is a polytope P contained in C such that $d_{\mathcal{H}}(P, C) < \varepsilon$. An analysis of the proof of this density statement gives the following variation which shows that *symmetric* convex bodies can be approximated by *symmetric* polytopes.

Lemma 2.2.6

If C is a symmetric convex body then for any $\varepsilon > 0$ there is a symmetric polytope P such that $d_{\mathcal{H}}(P, C) < \varepsilon$.

Proof: The compactness of C implies that for any $\varepsilon > 0$ there is a finite set of points $F' := \{x_1, x_2, \ldots, x_N\} \subset C$ such that $C \subset \bigcup_{i=1}^N B_\varepsilon(x_i) = F' + \varepsilon B$. Define $F := F' \cup (-F')$. (In the original proof Thompson only chooses $F := F'$.) Since C is symmetric, the extended set F is contained in C as well. Of course, $C \subset F + \varepsilon B$. Define the centrally symmetric polytope $P := \operatorname{conv}(F)$. Since C is convex and taking the convex hull preserves inclusions, the sequence of set inclusions

$$P \subset \operatorname{conv}(C) = C \subset F + \varepsilon B \subset P + \varepsilon B$$

holds. Hence, $\sup_{p\in P} \inf_{c\in C} \|p - c\| = 0$ by the first inclusion and $\sup_{c\in C} \inf_{p\in P} \|p - c\| < \varepsilon$ by the last. Therefore, P is a centrally symmetric polytope such that $d_{\mathcal{H}}(P, C) < \varepsilon$. $\square$

If we apply the previous result to the convex body B for $\varepsilon_l := \frac{1}{l}$, we may choose a sequence of symmetric polytopes $(B_l)_{l\in\mathbb{N}} \subset C_b$ such that $B_l \xrightarrow[l\to\infty]{} B$ in $(C_b, d_{\mathcal{H}})$. These polytopes are themselves convex sets by definition of a polytope. Without loss of generality, we assume that each B_l is a convex set of full dimension n (as defined in Section 1.2.1). Then each line through 0 meets B_l in a non-trivial (because 0 is an interior point of B_l), closed and bounded segment (since B_l is closed and bounded being a polytope, see Lemma 1.2.3). Then the ***Minkowski functional*** of each polytope B_l defined by

$$\|x\|_{B_l} := \inf\ \{ t \in \mathbb{R}_+ \mid x \in tB_l \}, x \in V$$

is a norm on V (see [Tho96, Theorem 1.1.8, p. 17]). In addition, B_l is the closed $\|\cdot\|_{B_l}$-unit ball. To see the latter, let $x \in V$ and $\|x\|_{B_l} \le 1$. Then by definition of the Minkowski functional $x \in tB_l \subset B_l$ where $t \le 1$ is the infimum in the definition. Conversely, if $x \in B_l$ then $\|x\|_{B_l} \le 1$.

Now we define the sequence of densities $\left(A_l^{bh}\right)_{l\in\mathbb{N}}$ where each A_l^{bh} is the Busemann–Hausdorff area density on the normed space $\left(V, \|\cdot\|_{B_l}\right)$ as defined by (2.1.8). Recall that the linear map $L = L_\sigma \colon \mathbb{R}^m \to V, Le_\alpha = v_\alpha$ for $\alpha = 1, 2, \ldots, m$ from the definition of the Busemann–Hausdorff area density is an injective linear map (or bijective onto its image). The intersection $B_l \cap \operatorname{im}(L)$ is a symmetric polytope of dimension m because $\operatorname{im}(L) = \operatorname{span}\{v_1, v_2, \ldots, v_m\}$ is an m-dimensional plane. Due to Lemma 1.2.16 (ii) we therefore know that the preimage $L^{-1}(B_l) = L^{-1}(B_l \cap \operatorname{im}(L))$ is a convex body. Notice that if $B_l \xrightarrow[l\to\infty]{} B$ in $(C_b(V), d_{\mathcal{H},V})$ then $L^{-1}(B_l) \xrightarrow[l\to\infty]{} L^{-1}(B)$ in $(C_b(\mathbb{R}^m), d_{\mathcal{H},\mathbb{R}^m})$ because

$$\begin{aligned}
d_{\mathcal{H},\mathbb{R}^m}&(L^{-1}(B_l),L^{-1}(B)) \\
&= \max\left(\sup_{x_1\in L^{-1}(B_l)} \inf_{x_2\in L^{-1}(B)} \|x_1 - x_2\|_{\mathbb{R}^m}, \sup_{x_2\in L^{-1}(B)} \inf_{x_1\in L^{-1}(B_l)} \|x_1 - x_2\|_{\mathbb{R}^m} \right) \\
&= \max\left(\sup_{y_1\in B_l} \inf_{y_2\in B} \left\|L^{-1}(y_1 - y_2)\right\|_{\mathbb{R}^m}, \sup_{y_2\in B} \inf_{y_1\in B_l} \left\|L^{-1}(y_1 - y_2)\right\|_{\mathbb{R}^m} \right) \\
&\le \left\|L^{-1}\right\|_{\mathcal{L}(\operatorname{im}(L),\mathbb{R}^m)} \max\left(\sup_{y_1\in B_l} \inf_{y_2\in B} \|y_1 - y_2\|_V, \sup_{y_2\in B} \inf_{y_1\in B_l} \|y_1 - y_2\|_V \right)
\end{aligned}$$

$$= \left\|L^{-1}\right\|_{\mathcal{L}(\mathrm{im}\,(L),\mathbb{R}^m)} d_{\mathcal{H},V}(B_l, B) \xrightarrow[l\to\infty]{} 0.$$

As mentioned earlier, the Lebesgue measure $\mathcal{L}^m = \mathcal{H}^m$ is continuous on the collection of compact convex sets in $\mathbb{R}^m$ with respect to the Hausdorff metric (see [Val68, Satz 12.7, p. 153] or [Bee74, p. 64]). Thus, for any $\sigma \in GC_m(V)$ we obtain

$$A_l^{bh}(\sigma) = \frac{\varepsilon_m}{\mathcal{H}^m(L_\sigma^{-1}(B_l))} \xrightarrow[l\to\infty]{} \frac{\varepsilon_m}{\mathcal{H}^m(L_\sigma^{-1}(B))} = A^{bh}(\sigma) \tag{2.2.18}$$

as l tends to infinity.

Let us assume that the convexity of the densities A_l^{bh} has already been proved. With (2.2.18) at hand we will conclude the convexity of the original Busemann–Hausdorff area density A^{bh} by using Lemma 2.2.4. Let $P \in G_m(V)$ be a fixed plane in V and $\sigma \in GC_m(V)$ an arbitrary simple m-vector. Then the convexity of each A_l^{bh} yields a sequence of calibrating m-forms $(w_l)_{l\in\mathbb{N}} \subset (\bigwedge^m(V))^*$ for P such that

$$|\omega_l(\sigma)| \leq A_l^{bh}(\sigma) \tag{2.2.19}$$

with equality if $\sigma \in GC_m(V) \cap \bigwedge^m(P)$. The convergence of the sequence $\left(A_l^{bh}(\sigma)\right)_{l\in\mathbb{N}}$ with limit $A^{bh}(\sigma)$ has been established in (2.2.18). Hence, we find a number $N \in \mathbb{N}$ such that

$$|\omega_l(\sigma)| \leq A_l^{bh}(\sigma) < A^{bh}(\sigma) + 1$$

for all $l > N$. Therefore, the sequence $(w_l(\sigma))_{l\in\mathbb{N}}$ is bounded in $\mathbb{R}$ and by sequential compactness there is a convergent subsequence $(w_{l_k}(\sigma))_{k\in\mathbb{N}}$. Passing to subsequences and renaming them, we find that for fixed $\sigma \in \bigwedge^m V$ both $\left(A_l^{bh}(\sigma)\right)_{l\in\mathbb{N}}$ and $(\omega_l(\sigma))_{l\in\mathbb{N}}$ converge and fulfil (2.2.19). Denote the limit of the latter sequence by $\omega(\sigma) := \lim_{l\to\infty} \omega_l(\sigma)$. The assignment $\sigma \to \omega(\sigma)$ is an m-form because it is the pointwise limit of such forms. Indeed, it is the sought-after calibrating form for A^{bh} because it holds that

$$\begin{aligned} |\omega(\sigma)| = \lim_{l\to\infty} |\omega_l(\sigma)| &= \limsup_{l\to\infty} |\omega_l(\sigma)| \\ &\leq \limsup_{l\to\infty} A_l^{bh}(\sigma) = \lim_{l\to\infty} A_l^{bh}(\sigma) = A^{bh}(\sigma) \end{aligned}$$

with equality if $\sigma \in GC_m(V) \cap \bigwedge^m(P)$. This shows the convexity of the density A^{bh} corresponding to the initial normed space $(V, \|\cdot\|_V)$ under the assumption that the densities A_l^{bh} corresponding to the polyhedral unit balls B_l are convex already.

Henceforth, Assumption 2.2.5 shall hold. We leave the general argumentation behind and focus on the case $m = 2$. As Lemma 2.2.4 suggests, the convexity of the corresponding two-dimensional Busemann–Hausdorff area density will be proved by explicitly constructing appropriate calibrators for A^{bh} and $m = 2$. We will reformulate the inequality Lemma 2.2.4 for a calibrator and prove a result of convex geometry on the two-dimensional plane.

Let us fix a two-dimensional linear subspace $P \in G_2(V)$ with basis $\{w_1, w_2\}$ and consider its intersection $B \cap P$ with the unit ball. Since B is a symmetric polytope this intersection is a symmetric polygon (a two-dimensional polytope) with vertices $\hat{a}_1, \hat{a}_2, \ldots, \hat{a}_{2N} \in V$. Recall from Theorem 1.2.7 that there is a half-space representation so that $B \cap P$ is the intersection of finitely many half-spaces. Let $F^i \in V^*$ be the linear form that supports $B \cap P$ on the segment $[\hat{a}_i, \hat{a}_{i+1}] = \{\, t\hat{a}_{i_1} + (1-t)\hat{a}_i \mid t \in [0,1] \,\}$. This means after appropriate scaling that $F^i\big|_{B\cap P} \leq 1$ and $F^i\big|_{[\hat{a}_i, \hat{a}_{i+1}]} = 1$ (for this, we need that $0 \in \operatorname{int}(B \cap P)$, see (1.2.15)). Further, let $L_P \colon \mathbb{R}^2 \to V$ be the linear map that takes the standard basis $\{e_1, e_2\}$ of $\mathbb{R}^2$ to $\{w_1, w_2\}$, that is, $L_P e_i = w_i$. Endowed with the standard Euclidean inner product $\mathbb{R}^2$ is an inner product space and the standard basis is orthonormal with respect to this inner product. Henceforth, whenever we use the words volume form and orientation we implicitly mean the standard volume form $\omega_e = e_1^* \wedge e_2^*$ corresponding to the standard basis and its induced standard orientation as introduced in Section 1.1.4.

Consider the preimage $K_P := L_P^{-1}(B \cap P)$ of the unit ball B in $\mathbb{R}^2$. This set corresponds to the intersection $B \cap P$. Then K_P is a convex body due to Lemma 1.2.16 (ii). Furthermore, K_P is the intersection of the half-spaces $\left\{x \in \mathbb{R}^2 \mid f_P^i(x) \leq 1\right\}$ where $f_P^i := F^i \circ L_P \in \left(\mathbb{R}^2\right)^*$ and therefore a polytope in $\mathbb{R}^2$ (Lemma 1.2.16 (iii)). In addition, according to Lemma 1.2.16 (iv) K_P is symmetric because B is. Thus, $K_P = [a_1 a_2 \ldots a_{2N}] \subset \mathbb{R}^2$ is a symmetric polygon with vertices $a_i = a_i(P) \in \mathbb{R}^2$. Suppose that the vertices are enumerated counterclockwise so that the 2-vectors $a_i \wedge a_j$ are positively oriented with respect to the standard volume form $\omega_e = e_1^* \wedge e_2^*$ for $1 \leq i < j \leq N$.

Define a 2-form $\omega \in \bigwedge^2(V^*)$ by

$$\omega = \omega(P) := \pi \sum_{1 \leq i < j \leq N} p_i p_j \; F^i \wedge F^j. \tag{2.2.20}$$

and set the coefficients as the area ratio of the triangle $\Delta 0 a_i a_{i+1} = \Delta 0 a_i(P) a_{i+1}(P) := [0 a_i a_{i+1}]$ and the entire polygon K_P, namely

$$p_i = p_i(P) := 2 \frac{\mathcal{H}^2(\Delta 0 a_i a_{i+1})}{\mathcal{H}^2(K_P)} \in [0, 1]$$

for $i = 1, 2, \dots, N$. Note that by the symmetry of K_P

$$\mathcal{H}^2(K_P) = \sum_{i=1}^{2N} \mathcal{H}^2(\Delta 0 a_i a_{i+1}) = 2 \sum_{i=1}^{N} \mathcal{H}^2(\Delta 0 a_i a_{i+1})$$

and thus $\sum_{i=1}^{N} p_i = 1$.

We are going to prove that ω is a calibrator for P, that is, for all $v_1 \wedge v_2 \in GC_2(V)$ we need to show that

$$|\omega(v_1 \wedge v_2)| \leq A^{bh}(v_1 \wedge v_2) \tag{2.2.21}$$

with equality if $v_1 \wedge v_2 \in GC_2(V) \cap \bigwedge^2(P)$.

Consider a simple 2-vector $\sigma = v_1 \wedge v_2 \in GC_2(V)$ where $v_1, v_2 \in V$ are linearly independent. Denote the two-dimensional plane spanned by these two vectors by $Q := \operatorname{span}\{v_1, v_2\}$. As σ runs through $GC_2(V)$ the corresponding plane runs through $G_2(V)$ (once again, this uses the injectivity of the Plücker embedding). The top exterior power $\bigwedge^2(P)$ of the two-dimensional subspace P is one-dimensional and therefore spanned by $w_1 \wedge w_2$. Thus, $v_1 \wedge v_2 \in GC_2(V) \cap \bigwedge^2(P)$ if and only if $v_1 \wedge v_2 = c w_1 \wedge w_2$. Recall from Section 1.1.3 on the Plücker embedding $\rho\colon GC_2(V) \to \mathbb{P}\left(\bigwedge^2(V)\right)$ that the latter means

$$\rho(Q) = \rho(\operatorname{span}\{v_1, v_2\}) = [v_1 \wedge v_2]_\sim = [w_1 \wedge w_2]_\sim = \rho(\operatorname{span}\{w_1, w_2\}) = \rho(P).$$

Since the Plücker embedding is injective by Proposition 1.1.9 we have the equivalence $v_1 \wedge v_2 \in GC_2(V) \cap \bigwedge^2(P)$ if and only if $Q = P$.

Corresponding to the plane Q let $L_Q\colon \mathbb{R}^2 \to V$ be the linear map such that the standard basis $\{e_1, e_2\}$ of $\mathbb{R}^2$ maps to $\{v_1, v_2\}$, that is, $L_Q e_i = v_i$. The left-hand side of (2.2.21) then reads

$$\begin{aligned}
|\omega(v_1 \wedge v_2)| &= |\omega(L_Q(e_1) \wedge L_Q(e_2))| \\
&= \pi \left| \sum_{1 \leq i < j \leq N} p_i p_j \left(F^i \wedge F^j\right)(L_Q(e_1) \wedge L_Q(e_2)) \right| \\
&= \pi \left| \sum_{1 \leq i < j \leq N} p_i p_j \left(\left(F^i \circ L_Q\right) \wedge \left(F^j \circ L_Q\right)\right)(e_1 \wedge e_2) \right|.
\end{aligned}$$

Define $f_Q^i := F^i \circ L_Q \in \left(\mathbb{R}^2\right)^*$. Any unit 2-vector σ in the top exterior power $\bigwedge^2(\mathbb{R}^2)$ is simple and either has a representation $\sigma = e_1 \wedge e_2$ or $\sigma = -e_1 \wedge e_2$ (see Corollary 1.1.12). Because of the absolute value and the linearity of $f_Q^i \wedge f_Q^j \in \bigwedge^2 \left((\mathbb{R}^2)^* \right) \cong \left(\bigwedge^2 (\mathbb{R}^2) \right)^*$ we can rewrite the last sum as

$$\begin{aligned}
|\omega(v_1 \wedge v_2)| &= \pi \sup \left\{ \left| \sum_{1 \le i < j \le N} p_i p_j \left(f_Q^i \wedge f_Q^j \right)(\sigma) \right| \,\middle|\, \sigma \in \{e_1 \wedge e_2, e_2 \wedge e_1\} \right\} \\
&= \pi \sup \left\{ \left| \sum_{1 \le i < j \le N} p_i p_j \left(f_Q^i \wedge f_Q^j \right)(\sigma) \right| \,\middle|\, \|\sigma\|_{\bigwedge^2(\mathbb{R}^2)} = 1 \right\} \\
&= \pi \left\| \sum_{1 \le i < j \le N} p_i p_j \; f_Q^i \wedge f_Q^j \right\|_{\left(\bigwedge^2(\mathbb{R}^2)\right)^*}
\end{aligned}$$

In the same way as earlier we can show that $K_Q := L_Q^{-1}(B \cap P)$ is a symmetric polygon in $\mathbb{R}^2$. By definition of the two-dimensional Busemann–Hausdorff area density in (2.1.8) the right-hand side of (2.2.21) reads

$$A^{bh}(v_1 \wedge v_2) = \frac{\varepsilon_2}{\mathcal{H}^2(L_Q^{-1}(B \cap P))} = \frac{\pi}{\mathcal{H}^2(K_Q)}.$$

Thus, to prove that $\omega(P)$ is a calibrator for the fixed plane P, we need to show for each plane $Q \in G_2(V)$ that

$$\left\| \sum_{1 \le i < j \le n} p_i(P) p_j(P) \; f_Q^i \wedge f_Q^j \right\|_{\left(\bigwedge^2(\mathbb{R}^2)\right)^*} \le \frac{1}{\mathcal{H}^2(K_Q)} \tag{2.2.22}$$

with equality if the plane Q coincides with the fixed plane P. Therein all dependencies on P and Q have been explicitly stated and the coefficients are given by

$$p_i(P) = 2 \frac{\mathcal{H}^2(\Delta 0 a_i(P) a_{i+1}(P))}{\mathcal{H}^2(K_P)}.$$

Note that the preceding discussion can similarly be made for arbitrary m. In fact, Burago and Ivanov give a characterisation for the convexity of the higher-dimensional Busemann–Hausdorff area density which is an m-dimensional analogue of (2.2.22) with coefficients $\mu_{i_1 i_2 \ldots i_m}$ on the left-hand side. The crux in the two-dimensional case is that one can choose $\mu_{ij} = p_i p_j$ so that the subsequent results hold. Burago and Ivanov mention that the most straight-forward generalisation of the two-dimensional construction (subdividing K_P into tetrahedrons instead of triangles) does not work, [BI12, Remark 4.2, p. 637].

To show (2.2.22) we prove the following more general statement from convex geometry on the plane.

Theorem 2.2.7 ([BI12, Proposition 2.2, p. 632])
Let $K \subset \mathbb{R}^2$ be a symmetric convex polygon and suppose $f^1, f^2, \ldots, f^N \in \left(\mathbb{R}^2\right)^*$ are linear forms such that $f^i\big|_K \leq 1$ for all $i = 1, 2, \ldots, N$ and $p_1, p_2, \ldots, p_N \in [0, 1]$ such that $\sum_{i=1}^N p_i = 1$. Then

$$\left\| \sum_{1 \leq i < j \leq N} p_i p_j \; f^i \wedge f^j \right\|_{\left(\bigwedge^2(\mathbb{R}^2)\right)^*} \leq \sum_{1 \leq i < j \leq N} p_i p_j \left\| f^i \wedge f^j \right\|_{\left(\bigwedge^2(\mathbb{R}^2)\right)^*} \leq \frac{1}{\mathcal{H}^2(K)}.$$

In addition, if K is a convex $2N$-gon with vertices $a_1, a_2, \ldots, a_{2N} \in \mathbb{R}^2$ and for $i = 1, 2, \ldots, N$ the functions f^i are supporting linear forms of K corresponding to its sides (such that $f^i\big|_{[a_i, a_{i+1}]} = 1$) and $p_i = 2\mathcal{H}^2(\Delta 0 a_i a_{i+1})/\mathcal{H}^2(K)$ where $\Delta 0 a_i a_{i+1} := [0 a_i a_{i+1}]$ denotes the triangle with vertices $0, a_i, a_{i+1}$, then the above inequalities turn into equalities.

The convexity of the two-dimensional Busemann–Hausdorff area density – Theorem 2.2.2 – is an immediate consequence of the previous result.

Proof of Theorem 2.2.2: Let $P \in G_2(V)$ be arbitrary but fixed. Due to the preceding discussion we need only show the inequality (2.2.22) for $Q \in G_2(V)$ where equality holds if $Q = P$. As previously shown K_Q is a symmetric $2N$-gon with vertices $a_i = a_i(Q)$ and f_Q^i are supporting linear forms corresponding to its sides. Further, $\sum_{i=1}^N p_i(P) = 1$. Theorem 2.2.7 for $K = K_Q$ yields the desired inequality. If, in addition, $Q = P$ note that the coefficient

$$p_i(P) = 2\frac{\mathcal{H}^2(\Delta 0 a_i(P) a_{i+1}(P))}{\mathcal{H}^2(K_P)} = 2\frac{\mathcal{H}^2(\Delta 0 a_i a_{i+1})}{\mathcal{H}^2(K_Q)} = p_i(Q)$$

which is the equality case in Theorem 2.2.7. Thus, $\omega = \omega(P)$ as defined in (2.2.20) is a indeed a calibrator for P. Since $P \in G_2(V)$ was arbitrary, Lemma 2.2.4 finally shows that the two-dimensional Busemann–Hausdorff area density A^{bh} admits a convex extension. $\square$

In the rest of this chapter we will prove Theorem 2.2.7 through several elementary lemmata. The next result provides us with a technical identity for the area of a polygon.

Lemma 2.2.8 ([BI12, Lemma 2.3, p. 632])
Let $K = [a_1 a_2 \ldots a_{2N}]$ be a symmetric $2N$-gon in $\mathbb{R}^2$. For $i = 1, 2, \ldots, N$ define $v_i := a_{i+1} - a_i$. Then

$$\mathcal{H}^2(K) = \sum_{1 \leq i < j \leq N} \|v_i \wedge v_j\|_{\bigwedge^2(\mathbb{R}^2)} = \left\| \sum_{1 \leq i < j \leq N} v_i \wedge v_j \right\|_{\bigwedge^2(\mathbb{R}^2)}.$$

Proof: Note that the vertices of K are positively oriented. Then the convexity of the polygon K

implies that the vectors v_i – which correspond to the edges of K – are also positively oriented, that is, $\omega_e(v_i \wedge v_j) > 0$ for any $1 \le i < j \le N$.

Since K is symmetric we observe that $\mathcal{H}^2(K) = 2\mathcal{H}^2([a_1 a_2 \dots a_N])$. Let us denote by $\Delta a_1 a_j a_{j+1} := [a_1 a_j a_{j+1}]$ the triangle with vertices a_1, a_j, a_{j+1}. Note that the intersection of any two of such triangles is a set of two-dimensional Lebesgue measure zero. The polygon $[a_1 a_2 \dots a_N]$ is the union of all triangles $\Delta a_1 a_j a_{j+1}$ where $j = 2, 3, \dots, N$. Then by the countable additivity of Hausdorff measure

$$\mathcal{H}^2(K) = 2\mathcal{H}^2([a_1 a_2 \dots a_N]) = 2\sum_{j=2}^{N} \mathcal{H}^2(\Delta a_1 a_j a_{j+1}). \tag{2.2.23}$$

Observe further that the area of the triangle $\Delta a_1 a_j a_{j+1}$ can be expressed via the norm of the wedge product of its edges due to Proposition 1.1.15. Then

$$\mathcal{H}^2(\Delta a_1 a_j a_{j+1}) = \frac{1}{2}\|(a_j - a_1) \wedge (a_{j+1} - a_j)\|_{\bigwedge^2(\mathbb{R}^2)} = \frac{1}{2}\|(a_j - a_1) \wedge v_j\|_{\bigwedge^2(\mathbb{R}^2)}.$$

By telescoping $a_j - a_1 = \sum_{i=1}^{j-1} a_{i+1} - a_i = \sum_{i=1}^{j-1} v_i$ and

$$\mathcal{H}^2(\Delta a_1 a_j a_{j+1}) = \frac{1}{2}\|(a_j - a_1) \wedge v_j\|_{\bigwedge^2(\mathbb{R}^2)} = \frac{1}{2}\left\|\sum_{i=1}^{j-1} v_i \wedge v_j\right\|_{\bigwedge^2(\mathbb{R}^2)}.$$

We apply Corollary 1.1.20 to the preceding identity and substitute the result into (2.2.23) which finally yields

$$\begin{aligned}
\mathcal{H}^2(K) &= 2\sum_{j=2}^{N} \mathcal{H}^2(\Delta a_1 a_j a_{j+1}) = 2\sum_{j=2}^{N} \frac{1}{2}\left\|\sum_{i=1}^{j-1} v_i \wedge v_j\right\|_{\bigwedge^2(\mathbb{R}^2)} \\
&= \sum_{j=2}^{N}\sum_{i=1}^{j-1} \|v_i \wedge v_j\|_{\bigwedge^2(\mathbb{R}^2)} \\
&= \sum_{1 \le i < j \le N} \|v_i \wedge v_j\|_{\bigwedge^2(\mathbb{R}^2)}.
\end{aligned}$$

□

The following lemma takes care of the equality case in Theorem 2.2.7.

Lemma 2.2.9 ([BI12, Lemma 2.4, p. 633])

Let $K = [a_1 a_2 \dots a_{2N}]$ be a symmetric $2N$-gon in $\mathbb{R}^2$. For $i = 1, 2, \dots, N$ define $v_i := a_{i+1} - a_i$ and $p_i := 2\mathcal{H}^2(\Delta 0 a_i a_{i+1})/\mathcal{H}^2(K)$ and let $f^i \in (\mathbb{R}^2)^*$ be such that $f^i\big|_K \le 1$ and $f^i\big|_{[a_i, a_{i+1}]} = 1$. Then

$$p_i p_j \left\|f^i \wedge f^j\right\|_{\left(\bigwedge^2(\mathbb{R}^2)\right)^*} = \frac{1}{\left(\mathcal{H}^2(K)\right)^2}\|v_i \wedge v_j\|_{\bigwedge^2(\mathbb{R}^2)}$$

for all $i, j = 1, 2, \dots, N$ and therefore

$$\left\| \sum_{1 \le i < j \le N} p_i p_j \; f^i \wedge f^j \right\|_{\left(\bigwedge^2(\mathbb{R}^2)\right)^*} = \sum_{1 \le i < j \le N} p_i p_j \left\| f^i \wedge f^j \right\|_{\left(\bigwedge^2(\mathbb{R}^2)\right)^*} = \frac{1}{\mathcal{H}^2(K)}.$$

PROOF: Define $S_i := 2\mathcal{H}^2(\Delta 0 a_i a_{i+1}) > 0$. Then by Corollary 1.1.20 the area of the parallelogram spanned by a_i and a_{i+1} can be expressed via $S_i = \|a_i \wedge a_{i+1}\|_{\bigwedge^2(\mathbb{R}^2)}$. Of course, $p_i = S_i/\mathcal{H}^2(K)$. We recall from Section 1.1.4 that the standard area form $\omega = \omega_e$ induces a Riesz-type isomorphism $\iota_\omega \colon \mathbb{R}^2 \to \left(\mathbb{R}^2\right)^*$ given by (1.1.8). We will now show that $\iota_\omega(v_i) = S_i f^i$. Note that there is no implied summation on i in this formula. This is an equation in the dual space $\left(\mathbb{R}^2\right)^*$, so we need to show $\iota_\omega(v_i)(u) = \omega(u \wedge v_i) = S_i f^i(u)$ for all $u \in \mathbb{R}^2$. Moreover, it suffices to prove this for a basis of $\mathbb{R}^2$ because both sides of the equation are linear functions. Observe that since K is convex, a_i and $v_i = a_{i+1} - a_i$ are linearly independent vectors and hence form a basis of $\mathbb{R}^2$. Indeed, (again there is no implied summation over i here)

$$S_i f^i(v_i) = S_i f^i(a_{i+1} - a_i) = S_i(f^i(a_{i+1}) - f(a_i)) = S_i(1-1) = 0 = \omega(0) = \omega(v_i \wedge v_i)$$

because f^i is linear and $f^i\big|_{[a_i, a_{i+1}]} = 1$. Further, by (1.1.7) calculate

$$S_i f^i(a_i) = S_i = \|a_i \wedge a_{i+1}\|_{\bigwedge^2(\mathbb{R}^2)} = \omega(a_i \wedge a_{i+1}) = \omega(a_i \wedge (a_{i+1} - a_i)) = \omega(a_i \wedge v_i)$$

where the third equality uses that the 2-vectors $a_i \wedge a_{i+1}$ are positively oriented. The isometric property of ι_ω that was established in Proposition 1.1.18 yields

$$\begin{aligned} p_i p_j \left\| f^i \wedge f^j \right\|_{\left(\bigwedge^2(\mathbb{R}^2)\right)^*} &= \frac{1}{\left(\mathcal{H}^2(K)\right)^2} S_i S_j \left\| f^i \wedge f^j \right\|_{\left(\bigwedge^2(\mathbb{R}^2)\right)^*} \\ &= \frac{1}{\left(\mathcal{H}^2(K)\right)^2} \left\| (S_i f^i) \wedge (S_j f^j) \right\|_{\left(\bigwedge^2(\mathbb{R}^2)\right)^*} \\ &= \frac{1}{\left(\mathcal{H}^2(K)\right)^2} \left\| \iota_\omega(v_i) \wedge \iota_\omega(v_j) \right\|_{\left(\bigwedge^2(\mathbb{R}^2)\right)^*} \\ &= \frac{1}{\left(\mathcal{H}^2(K)\right)^2} \left\| v_i \wedge v_i \right\|_{\bigwedge^2(\mathbb{R}^2)}. \end{aligned}$$

The second identity follows from Lemma 2.2.8 because

$$\begin{aligned} \sum_{1 \le i < j \le N} p_i p_j \left\| f^i \wedge f^j \right\|_{\left(\bigwedge^2(\mathbb{R}^2)\right)^*} &= \frac{1}{\left(\mathcal{H}^2(K)\right)^2} \sum_{1 \le i < j \le N} \left\| v_i \wedge v_i \right\|_{\bigwedge^2(\mathbb{R}^2)} \\ &= \frac{1}{\left(\mathcal{H}^2(K)\right)^2} \mathcal{H}^2(K) \end{aligned}$$

$$= \frac{1}{\mathcal{H}^2(K)}.$$

Recall the Riesz-type isomorphism ι_ω induces a dual volume form ω^* given by (1.1.11). The edges v_i of the polygon K are consistently (positively) oriented with respect to ω and therefore the corresponding supporting linear forms f^i are consistently (positively) oriented with respect to ω^* because

$$\omega^*(f^i \wedge f^j) = \frac{1}{S_i S_j}\omega\left((\iota_\omega)^{-1}(S_i f^i) \wedge (\iota_\omega)^{-1}(S_j f^j)\right) = \frac{1}{S_i S_j}\omega(v_i \wedge v_j) > 0$$

for $1 \le i < j \le N$, wherein $S_i > 0$. Finally, by Corollary 1.1.20

$$\left\| \sum_{1 \le i < j \le N} p_i p_j \; f^i \wedge f^j \right\|_{\left(\bigwedge^2(\mathbb{R}^2)\right)^*} = \sum_{1 \le i < j \le N} p_i p_j \left\| f^i \wedge f^j \right\|_{\left(\bigwedge^2(\mathbb{R}^2)\right)^*} = \frac{1}{\mathcal{H}^2(K)}. \qquad \square$$

It remains to show the inequality part of Theorem 2.2.7. The next lemma takes care of the principal case where the linear forms f^i support the polygon K at its edges.

Lemma 2.2.10 ([BI12, Lemma 2.5, p. 633])
Let $K = [a_1 a_2 \dots a_{2N}]$ be a symmetric $2N$-gon in $\mathbb{R}^2$. For $i = 1, 2, \dots, N$ let $f^i \in \left(\mathbb{R}^2\right)^*$ be such that $f^i\big|_K \le 1$ and $f^i\big|_{[a_i, a_{i+1}]} = 1$. Let $p_1, p_2, \dots, p_N \in [0,1]$ be nonnegative real numbers such that $\sum_{i=1}^N p_i = 1$. Then

$$\left\| \sum_{1 \le i < j \le N} p_i p_j \; f^i \wedge f^j \right\|_{\left(\bigwedge^2(\mathbb{R}^2)\right)^*} = \sum_{1 \le i < j \le N} p_i p_j \left\| f^i \wedge f^j \right\|_{\left(\bigwedge^2(\mathbb{R}^2)\right)^*} \le \frac{1}{\mathcal{H}^2(K)}.$$

Proof: The first identity follows because all the 2-forms $f^i \wedge f^j$ are of the same orientation for $i < j$ by the same argumentation as used in the proof of Lemma 2.2.9. Again for $i = 1, 2, \dots, N$ define the vectors $v_i := a_{i+1} - a_i$ corresponding to the edges of K. Further, let $q_i := 2\mathcal{H}^2(\Delta 0 a_i a_{i+1})/\mathcal{H}^2(K)$ and $\lambda_i := p_i/q_i$ for $i = 1, 2, \dots, N$. Due to the first assertion of Lemma 2.2.9

$$q_i q_j \left\| f^i \wedge f^j \right\|_{\left(\bigwedge^2(\mathbb{R}^2)\right)^*} = \frac{1}{\left(\mathcal{H}^2(K)\right)^2} \left\| v_i \wedge v_j \right\|_{\bigwedge^2(\mathbb{R}^2)}.$$

Define the vectors $v_i' := \lambda_i v_i$. By Lemma 1.2.15 there is a symmetric $2N$-gon $K' = [a_1' a_2' \dots a_{2N}']$ such that $a_{i+1}' - a_i' = v_i'$ for $i = 1, 2, \dots, N$. Then

$$\sum_{1 \le i < j \le N} p_i p_j \left\| f^i \wedge f^j \right\|_{\left(\bigwedge^2(\mathbb{R}^2)\right)^*} = \sum_{1 \le i < j \le N} \lambda_i \lambda_j q_i q_j \left\| f^i \wedge f^j \right\|_{\left(\bigwedge^2(\mathbb{R}^2)\right)^*}$$

$$= \frac{1}{\left(\mathcal{H}^2(K)\right)^2} \sum_{1\le i<j\le N} \lambda_i\lambda_j \|v_i \wedge v_j\|_{\bigwedge^2(\mathbb{R}^2)}$$
$$= \frac{1}{\left(\mathcal{H}^2(K)\right)^2} \sum_{1\le i<j\le N} \left\|v_i' \wedge v_j'\right\|_{\bigwedge^2(\mathbb{R}^2)}.$$

Applying Lemma 2.2.8 to K' and substituting the result into the last equation yields

$$\sum_{1\le i<j\le N} p_i p_j \left\|f^i \wedge f^j\right\|_{\left(\bigwedge^2(\mathbb{R}^2)\right)^*} = \frac{\mathcal{H}^2(K')}{\left(\mathcal{H}^2(K)\right)^2}. \tag{2.2.24}$$

Note that the 1-dimensional Lebesgue measure of the edges satisfy

$$\mathcal{L}^1\left([a_i', a_{i+1}']\right) = \|v_i'\|_{\mathbb{R}^2} = \lambda_i \|v_i\|_{\mathbb{R}^2} = \lambda_i \mathcal{L}^1\left([a_i, a_{i+1}]\right)$$

by construction of K'. Consider the support function h_K of the convex set K. By virtue of Proposition 1.2.6 the value $h_i := h_K\left(\frac{f^i}{\|f^i\|_{(\mathbb{R}^2)^*}}\right)$ is the distance of $0 \in K$ to the edge $[a_i, a_{i+1}]$ corresponding to f^i. Then the two-dimensional Hausdorff measure of the triangle $\Delta 0 a_i a_{i+1}$ is given by

$$\mathcal{H}^2(\Delta 0 a_i a_{i+1}) = \frac{1}{2} h_i \mathcal{L}^1\left([a_i, a_{i+1}]\right).$$

Now we can calculate

$$\begin{aligned} 1 = \sum_{i=1}^N p_i = \sum_{i=1}^N \lambda_i q_i &= \frac{2}{\mathcal{H}^2(K)} \sum_{i=1}^N \lambda_i \mathcal{H}^2(\Delta 0 a_i a_{i+1}) \\ &= \frac{2}{\mathcal{H}^2(K)} \frac{1}{2} \sum_{i=1}^N \lambda_i h_i \mathcal{L}^1\left([a_i, a_{i+1}]\right) \\ &= \frac{1}{\mathcal{H}^2(K)} \frac{1}{2} \sum_{i=1}^{2N} h_i \mathcal{L}^1\left([a_i', a_{i+1}']\right) \end{aligned} \tag{2.2.25}$$

where we used the symmetry of K to extend the sum in the last equality. The explicit formula for mixed volume in Lemma 1.2.14 (use $A = K'$ and $K = K$) shows that

$$\frac{1}{2} \sum_{i=1}^{2N} h_i \mathcal{L}^1\left([a_i', a_{i+1}']\right) = \mathrm{V}\left(K', K\right).$$

The preceding identity (2.2.25) thus transforms to

$$\mathcal{H}^2(K) = \mathrm{V}\left(K', K\right).$$

Now we use the Minkowski inequality for convex bodies (Proposition 1.2.12) to see that

$$\left(\mathcal{H}^2(K)\right)^2 = \left(\mathrm{V}\left(K', K\right)\right)^2 \geq \mathcal{H}^2(K')\mathcal{H}^2(K)$$

or, equivalently, $\mathcal{H}^2(K) \geq \mathcal{H}^2(K')$. Using this estimate on (2.2.24) proves the assertion of the lemma.□

Finally, we can complete the proof of Theorem 2.2.7 which in turn implies the convexity of the two-dimensional Busemann–Hausdorff area density as already shown. Thus, what follows concludes this chapter.

PROOF OF THEOREM 2.2.7: By means of Lemma 2.2.9 and Lemma 2.2.10 it remains to show the inequality for the case where K is a symmetric polygon (not necessarily with $2N$ vertices) and the linear forms $f^1, f^2, \ldots, f^N \in \left(\mathbb{R}^2\right)^*$ are such that $f^i\big|_K \leq 1$. The triangle inequality for $\|\cdot\|_{\left(\bigwedge^2(\mathbb{R}^2)\right)^*}$ yields

$$\left\| \sum_{1\leq i<j\leq N} p_i p_j \; f^i \wedge f^j \right\|_{\left(\bigwedge^2(\mathbb{R}^2)\right)^*} \leq \sum_{1\leq i<j\leq N} p_i p_j \left\| f^i \wedge f^j \right\|_{\left(\bigwedge^2(\mathbb{R}^2)\right)^*}.$$

Thus, we need to show that

$$\sum_{1\leq i<j\leq N} p_i p_j \left\| f^i \wedge f^j \right\|_{\left(\bigwedge^2(\mathbb{R}^2)\right)^*} \leq \frac{1}{\mathcal{H}^2(K)}. \tag{2.2.26}$$

Consider the left hand side of (2.2.26) as a function in just one variable f^k with the remaining f^i for $i \neq k$ staying fixed and let us call this function G_k. Then G_k is a convex function because it is the sum of the norms of 2-forms. Recall the definition of the polar set K° from (1.2.16). Then $f^i \in K^\circ$. Furthermore, the polar set of a polygon is itself a polygon by Proposition 1.2.9. This means that G_k is a convex function mapping the polar polygon K° to $\mathbb{R}$. In addition, G_k is bounded above because K° is bounded as a polytope and for each of the summands

$$\begin{aligned}
\left\| f^i \wedge f^j \right\|^2_{\left(\bigwedge^2(\mathbb{R}^2)\right)^*} &= \det \begin{pmatrix} \langle f_i, f_i \rangle_{(\mathbb{R}^2)^*} & \langle f_i, f_j \rangle_{(\mathbb{R}^2)^*} \\ \langle f_j, f_i \rangle_{(\mathbb{R}^2)^*} & \langle f_j, f_j \rangle_{(\mathbb{R}^2)^*} \end{pmatrix} \\
&= \|f_i\|^2_{(\mathbb{R}^2)^*} \|f_j\|^2_{(\mathbb{R}^2)^*} - 2\langle f_i, f_j \rangle_{(\mathbb{R}^2)^*} \\
&\leq \|f_i\|^2_{(\mathbb{R}^2)^*} \|f_j\|^2_{(\mathbb{R}^2)^*} + 2\|f_i\|_{(\mathbb{R}^2)^*} \|f_j\|_{(\mathbb{R}^2)^*} \\
&= \left(\|f_i\|_{(\mathbb{R}^2)^*} + \|f_j\|_{(\mathbb{R}^2)^*}\right)^2 < (2R)^2
\end{aligned}$$

where $R > 0$ such that K° is contained in the ball $B_R(0) \subset (\mathbb{R}^2)^*$ of radius R. Proposition 1.2.18

shows that G_k attains its maximum on a vertex of K°. But by Lemma 1.2.10 the vertices of K° are the linear forms that support K at its edges. Thus, it suffices to show (2.2.26) for the case where the f^i are (possibly duplicate) linear forms supporting K at its edges. Now if any two of the functions f^i and f^j coincide (without loss of generality $f_1 = f_N$) the problem can be reduced to a smaller number of functions by proceeding as follows. Calculate on the left hand side of (2.2.26)

$$\begin{aligned}\sum_{1\le i<j\le N} p_i p_j \big\|f^i \wedge f^j\big\|_{(\bigwedge^2(\mathbb{R}^2))^*} &= \sum_{i=1}^{N-1}\sum_{j=i+1}^{N} p_i p_j \big\|f^i \wedge f^j\big\|_{(\bigwedge^2(\mathbb{R}^2))^*}\\ &= \sum_{j=2}^{N} p_1 p_j \big\|f^1 \wedge f^j\big\|_{(\bigwedge^2(\mathbb{R}^2))^*} + \sum_{i=2}^{N-1}\sum_{j=i+1}^{N} p_i p_j \big\|f^i \wedge f^j\big\|_{(\bigwedge^2(\mathbb{R}^2))^*}.\end{aligned}$$

By expanding the second sum and using $f^1 \wedge f^N = 0$ on the first sum this reduces to

$$\begin{aligned}\sum_{1\le i<j\le N} p_i p_j \big\|f^i \wedge f^j\big\|_{(\bigwedge^2(\mathbb{R}^2))^*} &= \sum_{j=2}^{N-1} p_1 p_j \big\|f^1 \wedge f^j\big\|_{(\bigwedge^2(\mathbb{R}^2))^*}\\ &\quad + \sum_{i=2}^{N-1}\left(\sum_{j=i+1}^{N-1} p_i p_j \big\|f^i \wedge f^j\big\|_{(\bigwedge^2(\mathbb{R}^2))^*} + p_i p_N \big\|f^i \wedge f^N\big\|_{(\bigwedge^2(\mathbb{R}^2))^*}\right)\\ &= \sum_{j=2}^{N-1} p_1 p_j \big\|f^1 \wedge f^j\big\|_{(\bigwedge^2(\mathbb{R}^2))^*} + \sum_{i=2}^{N-1} p_i p_N \big\|f^i \wedge f^N\big\|_{(\bigwedge^2(\mathbb{R}^2))^*}\\ &\quad + \sum_{i=2}^{N-2}\sum_{j=i+1}^{N-1} p_i p_j \big\|f^i \wedge f^j\big\|_{(\bigwedge^2(\mathbb{R}^2))^*}\end{aligned}$$

Again we can use that $f^1 = f^N$ and thus,

$$\big\|f^1 \wedge f^i\big\|_{(\bigwedge^2(\mathbb{R}^2))^*} = \big\|-f^i \wedge f^N\big\|_{(\bigwedge^2(\mathbb{R}^2))^*} = \big\|f^i \wedge f^N\big\|_{(\bigwedge^2(\mathbb{R}^2))^*}.$$

Renaming the indices in the first two sums then yields

$$\begin{aligned}\sum_{1\le i<j\le N} p_i p_j \big\|f^i \wedge f^j\big\|_{(\bigwedge^2(\mathbb{R}^2))^*} &= \sum_{j=2}^{N-1} (p_1 + p_N) p_j \big\|f^1 \wedge f^j\big\|_{(\bigwedge^2(\mathbb{R}^2))^*} + \sum_{i=2}^{N-2}\sum_{j=i+1}^{N-1} p_i p_j \big\|f^i \wedge f^j\big\|_{(\bigwedge^2(\mathbb{R}^2))^*}\\ &= \sum_{i=1}^{N-2}\sum_{j=i+1}^{N-1} p'_i p'_j \big\|f^i \wedge f^j\big\|_{(\bigwedge^2(\mathbb{R}^2))^*}\end{aligned}$$

where $p'_1 := p_1 + p_N$ and $p'_i := p_i$ for $i > 1$. That is, the problem can be reduced to a smaller number of functions by dropping f_N from the list of functions and replacing the set of coefficients by $p_1 + p_N, p_2, \dots, p_{N-1}$. Additionally, note that changing the sign of one of the functions f^i does not change the left hand side of (2.2.26). Therefore, it further suffices to assume that all the functions $\pm f_1, \pm f_2, \dots, \pm f_N$ are distinct.

If $N = 1$ then the left hand side of (2.2.26) equals zero and the inequality holds trivially. Therefore, suppose $N > 1$ and consider the polyhedral set given by the half-space representation

$$K' := \bigcap_{i=1}^{N} \{x \in \mathbb{R}^2 \mid |f^i(x)| \leq 1\}.$$

Since K is symmetric and $f^i\big|_K \leq 1$ by hypothesis, we know that $f^i(x) \leq 1$ and $-f^i(x) = f^i(-x) \leq 1$ for $i = 1, 2, \dots, N$ and any $x \in K$. Therefore, $|f^i(x)| \leq 1$ for $i = 1, 2, \dots, N$ and any $x \in K$. In other words, $K \subset K'$. Therefore,

$$\mathcal{H}^2(K) \leq \mathcal{H}^2(K'). \tag{2.2.27}$$

Observe that K' is a symmetric polyhedral set and that $\pm f_1, \pm f_2, \dots, \pm f_N$ are those linear forms which give the half-space representation. By our assumption they correspond to N distinct pairs of opposing supporting hyperplanes. Since $N \geq 2$ there are at least two distinct pairs of such opposing supporting hyperplanes. Every polyhedral set in the two-dimensional space $\mathbb{R}^2$ that is supported by at least two distinct pairs of opposing hyperplanes is consequently bounded. Thus, K' is a bounded, polyhedral set which means by Theorem 1.2.7 that K' is a polygon. In two dimensions the number of vertices and facets (which are its edges) coincide (see [Brø83, §16 Euler's Relation, Theorem 16.1, p. 98]). Thus, K' is a $2N$-gon. By definition of K' through its half-space representation, we see that f^i are linear forms that support K' at its edges. Then all the prerequisites to apply Lemma 2.2.10 to K' are fulfilled. Using this fact and (2.2.27) finally proves (2.2.26) because

$$\sum_{1 \leq i < j \leq N} p_i p_j \left\| f^i \wedge f^j \right\|_{(\Lambda^2(\mathbb{R}^2))^*} \leq \frac{1}{\mathcal{H}^2(K')} \leq \frac{1}{\mathcal{H}^2(K)}. \qquad \square$$

Chapter 3

The Plateau problem in arbitrary codimension in the Finsler setting

In this final chapter we formulate and solve the Plateau problem in n-dimensional Finsler space $(\mathbb{R}^n, F)$ for a reversible Finsler metric F. To achieve this goal, we first reference the work of Hildebrandt and von der Mosel in which they developed the theory of Cartan functionals. We state their result on the Plateau problem for Cartan integrands (Theorem 3.2.1). Finally, we solve the Plateau problem in the Finsler setting by identifying the Busemann–Hausdorff area integrand a_m^F as a Cartan integrand for $m = 2$. For that, we use the convexity of the area integrand which we established in Chapter 2. Further, we use a representation of the area integrand, found by Overath in [Ove13].

3.1 Formulation of the Plateau problem in Finsler space

In this section we formalise the Plateau problem which was mentioned in the introduction. Let us recall the question it poses:

Given a closed rectifiable Jordan curve Γ, is there a minimal surface spanned by Γ?

For the following, denote the two-dimensional Euclidean unit ball in $\mathbb{R}^2$ by B. Its boundary is the set of vectors of Euclidean unit length and we denote it by $\mathbb{S}^1$.

A ***curve*** in $\mathbb{R}^n$ is a continuous mapping $c\colon [a,b] \to \mathbb{R}^n$ of an interval $[a,b]$ into $\mathbb{R}^n$. If a curve is injective then we call it a ***simple curve***. A curve c is said to be a ***closed curve*** if $c(a) = c(b)$. We can reparametrise a curve to be a continuous mapping $c\colon \mathbb{S}^1 \to \mathbb{R}^n$ (Consider the map $c \circ \varphi^{-1}$ where $\varphi\colon [a,b] \to \mathbb{S}^1, t \mapsto \left(\cos\left(2\pi\frac{t-a}{b-a}\right), \sin\left(2\pi\frac{t-a}{b-a}\right)\right)$). The image $\Gamma = \operatorname{im}(c)$ of a simple closed curve $c\colon \mathbb{S}^1 \to \mathbb{R}^n$ is called a ***Jordan curve***. A ***rectifiable curve*** is a curve of finite ***Euclidean length***, that is,

$$\mathcal{L}^{|\cdot|}(c) = \int_{\mathbb{S}^1} |\dot{c}(t)| \, dt < \infty.$$

Suppose Γ is a closed Jordan curve in $\mathbb{R}^n$ and let $\varphi \colon \mathbb{S}^1 \to \Gamma$ be a homeomorphism from $\mathbb{S}^1$ onto Γ. Then a continuous mapping $\psi \colon \mathbb{S}^1 \to \Gamma$ from $\mathbb{S}^1$ onto Γ is said to be ***weakly monotonic*** if there is a non-decreasing continuous function $\tau \colon [0, 2\pi] \to \mathbb{R}$ with $\tau(0) = 0, \tau(2\pi) = 2\pi$ such that

$$\psi(\cos(\theta), \sin(\theta)) = \varphi(\cos(\tau(\theta)), \sin(\tau(\theta)))$$

for $0 \le \theta \le 2\pi$ (see [DHKW92, Definition 2, p. 231]).

We now define the class of admissible surfaces for the Plateau problem. Recall that every function $X \in W^{1,2}(B, \mathbb{R}^n)$ has a trace $X\big|_{\mathbb{S}^1}$ on the boundary $\mathbb{S}^1$ which is of class $L^2(\mathbb{S}^1, \mathbb{R}^n)$ (see [Eva10, Theorem 1, p. 272])

Given a closed Jordan curve Γ in $\mathbb{R}^n$, a mapping $X \colon B \to \mathbb{R}^n$ is said to be of class $\mathcal{C}(\Gamma)$ if and only if $X \in W^{1,2}(B, \mathbb{R}^n)$ and its trace $X\big|_{\mathbb{S}^1}$ can be represented by a weakly monotonic, continuous mapping $\varphi \colon \mathbb{S}^1 \to \Gamma$ of $\mathbb{S}^1$ onto Γ (which means, every $L^2(\mathbb{S}^1, \mathbb{R}^n)$-representative of $X\big|_{\mathbb{S}^1}$ coincides with φ except for a subset of zero one-dimensional Hausdorff measure). One can show that $\mathcal{C}(\Gamma)$ is non-empty if the Jordan curve Γ is rectifiable (see [DHKW92, pp. 233-234]).

Finally, we can formulate the Plateau problem for Finsler area mentioned above. Notice that for $n = 3$ this coincides with the Plateau problem in [OvdM14, Theorem 1.2, p. 278].

Theorem 3.1.1 (Plateau problem for Finsler area in arbitrary codimension)
Let F be a reversible Finsler metric on $\mathbb{R}^n$ and assume in addition that

$$0 < m_F := \inf_{\mathbb{R}^n \times \mathbb{S}^{n-1}} F(\cdot, \cdot) \le \sup_{\mathbb{R}^n \times \mathbb{S}^{n-1}} F(\cdot, \cdot) = \|F\|_{L^\infty(\mathbb{R}^n \times \mathbb{S}^{n-1})} =: M_F < \infty.$$

Then for any given rectifiable Jordan curve $\Gamma \subset \mathbb{R}^n$ there exists a surface $X \in \mathcal{C}(\Gamma)$, such that

$$\mathrm{area}_B^F(X) = \inf_{\mathcal{C}(\Gamma)} \mathrm{area}_B^F(\cdot).$$

In addition, the minimiser X is Euclidean conformally parametrised almost everywhere on B, that is,

$$\left|\frac{\partial X}{\partial u^1}\right|^2 = \left|\frac{\partial X}{\partial u^2}\right|^2 \qquad \text{and} \qquad \left\langle \frac{\partial X}{\partial u^1}, \frac{\partial X}{\partial u^2} \right\rangle_{\mathbb{R}^2} = 0 \quad \mathcal{H}^2\text{– a.e. on } B.$$

Furthermore, the minimiser X has the following regularity property,

$$X \in C^0\left(\overline{B}, \mathbb{R}^n\right) \cap C^{0,\sigma}\left(B, \mathbb{R}^n\right) \cap W^{1,q}\left(B, \mathbb{R}^n\right)$$

for $\sigma := \left(\frac{m_F}{M_F}\right)^2 \in (0,1]$ *and some* $q > 2$.

3.2 Cartan functional theory

In the work of Hildebrandt and von der Mosel [HvdM99, HvdM02, HvdM03a, HvdM03b, HvdM03c, HvdM05, HvdM09] the theory of Cartan (or parametric) functionals has been developed. In particular, the Plateau problem for Cartan integrands has been solved and higher regularity of the minimizers has been established. We want to apply their theory to the Busemann–Hausdorff area integrand and prove Theorem 3.1.1 from the preceding section.

First, let us define the basics of Cartan functional theory. For two integers n, m with $n \geq m$, set $N := \binom{n}{m}$. Recall from Corollary 1.1.13 that we can isometrically identify the spaces $\left(\bigwedge^m(\mathbb{R}^n), \|\cdot\|_{\bigwedge^m(\mathbb{R}^n)}\right)$ and $\left(\mathbb{R}^N, |\cdot|\right)$ where $|\cdot|$ is the standard Euclidean norm on $\mathbb{R}^N$. We denote the image of the set $GC_m(\mathbb{R}^n) \subset \bigwedge^m(\mathbb{R}^n)$ under this identification as $\mathbb{GC}_m(\mathbb{R}^n) \subset \mathbb{R}^N$.

A function $I\colon \mathbb{R}^n \times \mathbb{R}^N \to \mathbb{R}$ is called a ***Cartan integrand*** if it is continuous, that is,

$$I \in C^0\left(\mathbb{R}^n \times \mathbb{R}^N\right) \tag{R}$$

and if it is homogeneous of degree one in its second variable, that is,

$$I(x, tz) = tI(x, z) \quad \text{for all } t > 0, (x, z) \in \mathbb{R}^n \times \mathbb{R}^N. \tag{H}$$

A Cartan integrand I is said to be ***positive definite*** if there are two constants M_1 and M_2 with $0 < M_1 \leq M_2$ such that

$$M_1|z| \leq I(x, z) \leq M_2|z| \quad \text{for all } (x, z) \in \mathbb{R}^n \times \mathbb{GC}_m(\mathbb{R}^n). \tag{D}$$

A Cartan integrand I is said to be ***semi–elliptic*** on $\Omega \times \mathbb{R}^N$ for $\Omega \subset \mathbb{R}^n$ if it is convex in its second variable, that is, if

$$I(x, tz_1 + (1-t)z_2) \leq tI(x, z_1) + (1-t)I(x, z_2) \quad \text{for all } x \in \Omega, z_1, z_2 \in \mathbb{R}^N \text{ and } t \in [0, 1]. \tag{C}$$

Given a Cartan integrand I we can define the ***Cartan functional*** $\boldsymbol{\mathcal{I}}$ in the following way. Suppose $\mathcal{M}$ is a smooth m-manifold and $X\colon \mathcal{M} \to \mathbb{R}^n$ a smooth immersion. Then we define

$$\mathcal{I}(X) := \int_{p \in \mathcal{M}} i(p),$$

where the differential m-form i on $\mathcal{M}$ is given in local coordinates $(u^1, u^2, \ldots, u^m)\colon U \subset \mathcal{M} \to \Omega \subset \mathbb{R}^n$

by

$$i(p) := I\left(X(p), dX_p\left(\frac{\partial}{\partial u^1}\bigg|_p\right) \wedge dX_p\left(\frac{\partial}{\partial u^2}\bigg|_p\right) \wedge \cdots \wedge dX_p\left(\frac{\partial}{\partial u^m}\bigg|_p\right)\right).$$

This form is well-defined globally due to its invariance under a coordinate change ([Ove13, p. 47]). If we choose $\mathcal{M} = \Omega$ for an open subset $\Omega \subset \mathbb{R}^m$ and (u^α) to be the standard coordinates on $\mathbb{R}^m$, then $\mathcal{I}(X)$ computes to

$$\mathcal{I}(X) = \int_{u\in\Omega} I\left(X(u), \frac{\partial X}{\partial u^1}(u) \wedge \frac{\partial X}{\partial u^2}(u) \wedge \cdots \wedge \frac{\partial X}{\partial u^m}(u)\right) du^1 du^2 \ldots du^m.$$

The value $\mathcal{I}(X)$ can be guaranteed to be finite by assuming $X \in W^{1,m}(\Omega, \mathbb{R}^n)$ and I to satisfy the positive definiteness relation (D). We can see this by combining these assumptions, Lemma 1.1.14 and the identification $\mathbb{R}^N \cong \bigwedge^m(\mathbb{R}^n)$ to get the estimate

$$\begin{aligned} I\left(X(u), \frac{\partial X}{\partial u^1}(u) \wedge \frac{\partial X}{\partial u^2}(u) \wedge \cdots \wedge \frac{\partial X}{\partial u^m}(u)\right) &\leq M_2 \left|\frac{\partial X}{\partial u^1}(u) \wedge \frac{\partial X}{\partial u^2}(u) \wedge \cdots \wedge \frac{\partial X}{\partial u^m}(u)\right| \\ &= M_2 \left\|\frac{\partial X}{\partial u^1}(u) \wedge \frac{\partial X}{\partial u^2}(u) \wedge \cdots \wedge \frac{\partial X}{\partial u^m}(u)\right\|_{\bigwedge^m(\mathbb{R}^n)} \\ &\leq M_2 \prod_{i=1}^m \left\|\frac{\partial X}{\partial u^i}(u)\right\|_{\mathbb{R}^n} \\ &= M_2 \prod_{i=1}^m \left|\frac{\partial X}{\partial u^i}(u)\right|. \end{aligned}$$

On the last expression we can apply the inequality of arithmetic and geometric means and the Hölder inequality and get

$$M_2 \prod_{i=1}^m \left|\frac{\partial X}{\partial u^i}(u)\right| \leq M_2 \frac{1}{m^m} \left(\sum_{i=1}^m \left|\frac{\partial X}{\partial u^i}(u)\right|\right)^m \leq M_2 \frac{1}{m^m} m^{m-1} \sum_{i=1}^m \left|\frac{\partial X}{\partial u^i}(u)\right|^m.$$

Thus, it follows that

$$\begin{aligned} \mathcal{I}(X) &= \int_{u\in\Omega} I\left(X(u), \frac{\partial X}{\partial u^1}(u) \wedge \frac{\partial X}{\partial u^2}(u) \wedge \cdots \wedge \frac{\partial X}{\partial u^m}(u)\right) du^1 du^2 \ldots du^m \\ &\leq M_2 \frac{1}{m} \int_{u\in\Omega} \sum_{i=1}^m \left|\frac{\partial X}{\partial u^i}(u)\right|^m du^1 du^2 \ldots du^m \\ &\leq M_2 \frac{1}{m} \|X\|^m_{W^{1,m}(B,\mathbb{R}^n)} < \infty. \end{aligned}$$

Hildebrandt and von der Mosel proved the following result on the minimisation of the Cartan functional in the class of admissible surface $\mathcal{C}(\Gamma)$ introduced in Section 3.1. Note that therein we chose $\mathcal{M} = B$ as the parameter domain of the competing surfaces X. Further, $m = 2$ and $X \in W^{1,2}(B, \mathbb{R}^n)$ for $X \in \mathcal{C}(\Gamma)$ such that the value of the Cartan functional is finite by the preceding calculation.

Theorem 3.2.1 (Plateau problem for Cartan integrands, [HvdM03b, Theorems 1.4 and 1.5, p. 928]) *Suppose $I\colon \mathbb{R}^n \times \mathbb{R}^N \to \mathbb{R}$ satisfies (R), (H), (D) and (C). Then for any given rectifiable Jordan curve $\Gamma \subset \mathbb{R}^n$ there exists a surface $X \in \mathcal{C}(\Gamma)$, such that*

$$\mathcal{I}(X) = \inf_{\mathcal{C}(\Gamma)} \mathcal{I}(\cdot).$$

In addition, the minimiser X is Euclidean conformally parametrised almost everywhere on B, that is,

$$\left|\frac{\partial X}{\partial u^1}\right|^2 = \left|\frac{\partial X}{\partial u^2}\right|^2 \qquad \textit{and} \qquad \left\langle \frac{\partial X}{\partial u^1}, \frac{\partial X}{\partial u^2} \right\rangle_{\mathbb{R}^2} = 0 \quad \mathcal{H}^2\text{– a.e. on } B.$$

Furthermore, the minimiser X has the following regularity property,

$$X \in C^0\left(\overline{B}, \mathbb{R}^n\right) \cap C^{0,\sigma}\left(B, \mathbb{R}^n\right) \cap W^{1,q}\left(B, \mathbb{R}^n\right)$$

for $\sigma := \left(\frac{m_F}{M_F}\right)^2 \in (0, 1]$ and some $q > 2$.

Note that in [HvdM03b] a Cartan integrand is called positive definite if the inequalities in (D) hold for all of $\mathbb{R}^n \times \mathbb{R}^N$. However, an analysis of their proof shows that condition (D) as stated above suffices to prove Theorem 3.2.1.

3.3 Representing the Busemann–Hausdorff area as a Cartan functional

To prove Theorem 3.1.1, we wish to identify the two-dimensional Busemann–Hausdorff area integrand a_2^F for a reversible Finsler metric on $\mathcal{N} = \mathbb{R}^n$ as a positive definite, semi-elliptic Cartan integrand. Recall from (2.1.6) that a_m^F is the function

$$\begin{aligned} a_m^F \colon & \bigsqcup_{q\in\mathcal{N}} GC_m(T_q\mathbb{R}^n) \to \mathbb{R}_+, \\ & (q, w_1 \wedge w_2 \wedge \cdots \wedge w_m) \mapsto \frac{\varepsilon_m}{\mathcal{H}^m\left(\{v \in \mathbb{R}^m \mid F\left(q, v^\alpha w_\alpha\right) \le 1\}\right)} \end{aligned}$$

where we sum over Greek indices $\alpha = 1, 2, \dots, m$.

As discussed in the previous chapter, the tangent spaces $T_q\mathbb{R}^n$ are canonically isomorphic to $\mathbb{R}^n$ itself and we mentioned above that $\bigwedge^m(T_q\mathbb{R}^n) \cong \bigwedge^m(\mathbb{R}^n) \cong \mathbb{R}^N$. So we can identify the disjoint union as

$$\bigsqcup_{q\in\mathbb{R}^n} \bigwedge^m (T_q\mathbb{R}^n) \cong \bigsqcup_{q\in\mathbb{R}^n} \mathbb{R}^N = \bigcup_{q\in\mathbb{R}^n} \{q\} \times \mathbb{R}^N = \mathbb{R}^n \times \mathbb{R}^N. \tag{3.3.1}$$

One can show that $\bigsqcup_{q\in\mathcal{N}} \bigwedge^m(T_q\mathcal{N})$ is a ***vector bundle of rank N*** (see [Lee13, Exercise 14.14, p. 359]). In fact, if $\mathcal{N} = \mathbb{R}^n$ then it is a trivial bundle which means that the identification (3.3.1) is a homeomorphism. Under this identification, the Busemann–Hausdorff area integrand a_m^F is a function mapping the subset $\mathbb{R}^n \times \mathbb{GC}_m(\mathbb{R}^n) \subset \mathbb{R}^n \times \mathbb{R}^N$ to $\mathbb{R}_+$.

Recall as well that by (2.1.9)

$$a_m^F(q, v_1 \wedge v_2 \wedge \cdots \wedge v_m) = A_{q,m}^{bh}(v_1 \wedge v_2 \wedge \cdots \wedge v_m)$$

for $q \in \mathbb{R}^n$ and $v_1 \wedge v_2 \wedge \cdots \wedge v_m \in GC_m(\mathbb{R}^n)$. Therein, $A^{bh} = A_{q,m}^{bh}$ was the m-dimensional Busemann–Hausdorff area density for the normed space $(\mathbb{R}^n, F(q,\cdot)) \cong (T_q\mathbb{R}^n, F(q,\cdot))$.

3.3.1 Semi–ellipticity and homogeneity of $\mathfrak{a}_2^F$ on $\mathbb{R}^n \times \mathbb{R}^N$

For $m = 2$ (and thus $N = \binom{n}{2}$) we showed in Theorem 2.2.2 that the two-dimensional Busemann–Hausdorff area density $A_{q,2}^{bh} \colon GC_2(\mathbb{R}^n) \cong \mathbb{GC}_2(\mathbb{R}^n) \to \mathbb{R}_+$ is convex, that is, it admits an absolutely homogeneous, continuous and convex extension $\mathcal{A} = \mathcal{A}_{q,2} \colon \bigwedge^2(\mathbb{R}^n) \cong \mathbb{R}^N \to \mathbb{R}_+$ (see Definition 2.2.1)

Therefore, the two-dimensional Busemann–Hausdorff area integrand extends to a function

$$\mathfrak{a}_2^F : \mathbb{R}^n \times \mathbb{R}^N \to \mathbb{R}_+, (q, \sigma) \mapsto \mathcal{A}_{q,2}(\sigma)$$

which we call the ***extended two-dimensional Busemann–Hausdorff area integrand***.

Since $\mathcal{A}$ is convex for every normed space $(\mathbb{R}^n, F(q,\cdot))$, the function $\mathfrak{a}_2^F$ is convex in its second variable and therefore the extended two-dimensional Busemann–Hausdorff area integrand is semi-elliptic on $\mathbb{R}^n \times \mathbb{R}^N$, that is, (C) holds. Further, $\mathfrak{a}_2^F$ is homogeneous of degree one in its second variable because $\mathcal{A}$ is and so, (H) holds.

3.3.2 Continuity of $\mathfrak{a}_2^F$ on $\mathbb{R}^n \times \mathbb{R}^N$

For the regularity of $\mathfrak{a}_2^F$, we consider the extended Busemann–Hausdorff area integrand as a mapping from $\bigsqcup_{q\in\mathbb{R}^n} \bigwedge^2(T_q\mathbb{R}^n)$ to $\mathbb{R}$.

Theorem 3.3.1 (Continuity of $\boldsymbol{\mathfrak{a}_2^F}$)
The extended two-dimensional Busemann–Hausdorff area integrand $\mathfrak{a}_2^F : \bigsqcup_{q\in\mathbb{R}^n} \bigwedge^2(T_q\mathbb{R}^n) \to \mathbb{R}_+$ *is a continuous function. Therefore, by using the identification* (3.3.1)

$$\mathfrak{a}_2^F \in C^0(\mathbb{R}^n \times \mathbb{R}^N),$$

that is, the condition (R) *holds true.*

Proof: We recall that $\mathfrak{a}_2^F(q,\sigma) = \mathcal{A}_{q,2}(\sigma)$ for $(q,\sigma) \in \bigsqcup_{q\in\mathbb{R}^n} \bigwedge^2(T_q\mathbb{R}^n)$. Each of the extensions $\mathcal{A}_{q,2}$ is a continuous function by Theorem 2.2.2 and Definition 2.2.1. Since $\mathbb{R}^n$ is a smooth manifold, we know that its tangent spaces $T_q\mathbb{R}^n$ vary continuously in q. Thus, $\mathfrak{a}_2^F$ is also continuous in its first variable. Using local trivialisations of the vector bundle $\bigsqcup_{q\in\mathbb{R}^n} \bigwedge^2(T_q\mathbb{R}^n)$, it then follows that $\mathfrak{a}_2^F$ is continuous on all of $\bigsqcup_{q\in\mathbb{R}^n} \bigwedge^2(T_q\mathbb{R}^n)$. □

3.3.3 Positive definiteness of $\mathfrak{a}_2^F$ on $\mathbb{R}^n \times \mathbb{R}^N$

Condition (D) is only a condition on $\mathbb{R}^n \times \mathbb{GC}_m(\mathbb{R}^n)$. Therefore, we need only show it for the Busemann–Hausdorff area integrand and not for its extension. In fact, (D) holds true for a_m^F for arbitrary m as we will see in the following. In the work of Overath [Ove13] it was found that the Busemann–Hausdorff area integrand can be represented as a spherical integral. This result will provide us with a way to quantify the positive definiteness of a_m^F.

Theorem 3.3.2 (Spherical integral representation of $\boldsymbol{a_m^F}$, [Ove13, Theorem 2.1.6, p. 70])
Let $n \geq m > 0$, $(\mathcal{N}^n, F)$ be a Finsler manifold where F is strictly positive on $T\mathcal{N} \setminus o$. Let $q \in \mathcal{N}$ and

$w_1 \wedge w_2 \wedge \ldots \wedge w_m \in GC_m(T_q\mathcal{N})$. *Then we have*

$$a_m^F(q, w_1 \wedge w_2 \wedge \cdots \wedge w_m) = \left(\frac{1}{\mathcal{H}^{m-1}(\mathbb{S}^{m-1})} \int_{\mathbb{S}^{m-1}} \frac{1}{(F(q, \theta^\alpha w_\alpha))^m} dS(\theta) \right)^{-1}. \tag{3.3.2}$$

Therein $dS(\theta)$ is the ***spherical measure*** *and $\mathbb{S}^{m-1} = \{x \in \mathbb{R}^m \mid \; |x| = 1\}$ is the m-dimensional Euclidean unit sphere.*

Proof: We need only rewrite the Hausdorff measure in expression (2.1.6) by a change of variables (see [For09, Satz 8, p. 144]). Using polar coordinates we can write any $v = (v^\alpha)_{\alpha=1}^m \in \mathbb{R}^m$ as $v = s\theta$ where $\theta = v/|v| \in \mathbb{S}^{m-1}$ and $s = |v| \in [0, \infty)$. Then

$$\begin{aligned}
&\mathcal{H}^m \left(\{ v \in \mathbb{R}^m \mid F(q, v^\alpha w_\alpha) \le 1 \} \right) \\
&= \int_{\mathbb{R}^m} \chi_{\{v \in \mathbb{R}^m \mid F(q, v^\alpha w_\alpha) \le 1\}}(x) d\mathcal{L}^m(x) \\
&= \int_{\mathbb{S}^{m-1}} \int_0^\infty \chi_{\left\{(r\theta') \in \mathbb{R}^m \;\middle|\; F(q, r\theta'^\alpha w_\alpha) \le 1\right\}}(s\theta) s^{m-1} ds dS(\theta)
\end{aligned}$$

wherein χ_M is the ***characteristic function of M***, that is, $\chi_M(z) := 1$ if $z \in M$ and 0 otherwise. The Finsler metric F is positive homogeneous in its second component by definition. Further, F is strictly positive on $T\mathcal{N} \setminus o$. Therefore,

$$\begin{aligned}
&\int_{\mathbb{S}^{m-1}} \int_0^\infty \chi_{\left\{(r\theta') \in \mathbb{R}^m \;\middle|\; F(q, r\theta'^\alpha w_\alpha) \le 1\right\}}(s\theta) s^{m-1} ds dS(\theta) \\
&= \int_{\mathbb{S}^{m-1}} \int_0^\infty \chi_{\left\{(r\theta') \in \mathbb{R}^m \;\middle|\; r \le (F(q, \theta'^\alpha w_\alpha))^{-1}\right\}}(s\theta) s^{m-1} ds dS(\theta) \\
&= \int_{\mathbb{S}^{m-1}} \int_0^{(F(q, \theta^\alpha w_\alpha))^{-1}} s^{m-1} ds dS(\theta) \\
&= \int_{\mathbb{S}^{m-1}} \frac{1}{m \left(F \left(q, \theta^\alpha w_\alpha \right) \right)^m} dS(\theta).
\end{aligned}$$

Substituting into (2.1.6) and using $\mathcal{H}^{m-1}(\mathbb{S}^{m-1}) = m\mathcal{H}^m(B_1^m(0)) = m\varepsilon_m$, we find

$$\begin{aligned}
a_m^F(q, w_1 \wedge w_2 \wedge \cdots \wedge w_m) &= \frac{\varepsilon_m}{\int_{\mathbb{S}^{m-1}} \frac{1}{m(F(q, \theta^\alpha w_\alpha))^m} dS(\theta)} \\
&= \left(\frac{1}{\mathcal{H}^{m-1}(\mathbb{S}^{m-1})} \int_{\mathbb{S}^{m-1}} \frac{1}{(F(q, \theta^\alpha w_\alpha))^m} dS(\theta) \right)^{-1}. \qquad \square
\end{aligned}$$

The next result shows that the Busemann–Hausdorff area integrand coincides with the classical area integrand in the case that the Finsler metric is the Euclidean norm.

Lemma 3.3.3 (Euclidean area integrand)

Define $F\colon \mathbb{R}^n \times \mathbb{R}^n \to R_+$ by $F(q,v) := |v|$ where $|\cdot|$ is the Euclidean norm on $\mathbb{R}^n$. Then F is a Finsler metric on $\mathbb{R}^n$ which is strictly positive on $\mathbb{R}^n \times (\mathbb{R}^n \setminus \{0\})$ and

$$a_m^{|\cdot|}(q,\sigma) = |\sigma| \quad \textit{for all } (q,\sigma) \in \mathbb{R}^n \times \mathbb{GC}_m(\mathbb{R}^n) \tag{3.3.3}$$

where $|\cdot|$ denotes the n-dimensional Euclidean norm on the left-hand side and the $\binom{n}{m}$-dimensional one on the right-hand side.

Proof: Clearly, F is non-negative on $\mathbb{R}^n \times \mathbb{R}^n$, strictly positive on $\mathbb{R}^n \times (\mathbb{R}^n \setminus \{0\})$ and positively homogeneous in its second argument. In addition, $F \in C^\infty(\mathbb{R}^n \times (\mathbb{R}^n \setminus \{0\})) \cap C^0(\mathbb{R}^n \times \mathbb{R}^n)$. Note that

$$F(q,v) = \left(\delta_{ij} v^i v^j\right)^{\frac{1}{2}}$$

where δ_{ij} is the Kronecker delta. Then the numbers

$$g_{ij}(q,v) = \frac{1}{2}\frac{\partial^2}{\partial v^i \partial v^j} F^2(q,v) = \frac{1}{2}\delta_{ij}$$

form a positive definite matrix, so that the ellipticity condition for F is fulfilled.

To prove the second assertion, we follow the calculation in [Ove13, pp. 72-73]. The goal is to apply the area formula [EG92, Section 3.3.2, Theorem 1] to the denominator in (2.1.6). Fix $q \in \mathbb{R}^n$ and $\sigma \in GC_m(\mathbb{R}^n)$ and suppose $\sigma = w_1 \wedge w_2 \wedge \cdots \wedge w_m$ where $w_1, w_2, \cdots, w_m \in \mathbb{R}^n$ are linearly independent.

We will prove

$$\varepsilon_m = \|\sigma\|_{\bigwedge^m(\mathbb{R}^n)} \mathcal{H}^m\left(\{v \in \mathbb{R}^m \mid |v^\alpha w_\alpha| \le 1\}\right). \tag{3.3.4}$$

wherewith it follows that

$$a_m^{|\cdot|}(q,\sigma) = \|\sigma\|_{\bigwedge^m(\mathbb{R}^n)}.$$

As mentioned in the beginning of the present section, we recall from Corollary 1.1.13 that the spaces $\left(\bigwedge^m(\mathbb{R}^n), \|\cdot\|_{\bigwedge^m(\mathbb{R}^n)}\right)$ and $\left(\mathbb{R}^N, |\cdot|\right)$ are isometrically isomorph. So

$$\|\sigma\|_{\bigwedge^m(\mathbb{R}^n)} = |\sigma|$$

where we use the same symbol σ for an element of $GC_m(\mathbb{R}^n) \subset \bigwedge^m(\mathbb{R}^n)$ and of $\mathbb{GC}_m(\mathbb{R}^n) \subset \mathbb{R}^N$ on the left- and right-hand side respectively. Combining this with the above, the assertion holds true.

To prove (3.3.4) define the map $f\colon \mathbb{R}^m \to \mathbb{R}^n, v \mapsto v^\alpha w_\alpha$. This is an injective linear mapping with $\mathrm{im}\,(f) = \mathrm{span}\,\{w_1, w_2, \ldots, w_m\} \subset \mathbb{R}^n$. Note that the ***Jacobian matrix*** of f is

$$\begin{aligned} Df(v) &:= \begin{pmatrix} \frac{\partial f^1}{\partial v^1} & \cdots & \frac{\partial f^1}{\partial v^m} \\ \vdots & \ddots & \vdots \\ \frac{\partial f^n}{\partial v^1} & \cdots & \frac{\partial f^n}{\partial v^m} \end{pmatrix} \\ &= \big(w_1 | w_2 | \cdots | w_m\big) \in \mathbb{R}^{n\times m} \end{aligned}$$

and the ***Jacobian determinant*** of f is

$$\begin{aligned} Jf(v) &:= \sqrt{\det\left(Df(v)^T Df(v)\right)} \\ &= \sqrt{\det\left(\left(\langle w_\alpha, w_\beta\rangle_{\mathbb{R}^n}\right)_{\alpha,\beta=1,\ldots,m}\right)} \\ &= \|w_1 \wedge w_2 \wedge \cdots \wedge w_m\|_{\bigwedge^m(\mathbb{R}^n)} = \|\sigma\|_{\bigwedge^m(\mathbb{R}^n)}. \end{aligned}$$

Define the two sets

$$\begin{aligned} A &:= \{v \in \mathbb{R}^m \mid |v^\alpha w_\alpha| \le 1\} \subset \mathbb{R}^m \\ \Omega &:= \overline{B_1^n(0)} \cap \mathrm{span}\,\{w_1, w_2, \ldots, w_m\} \subset \mathbb{R}^n \end{aligned}$$

wherein $B_1^n(0) \subset \mathbb{R}^n$ denotes the n-dimensional Euclidean unit ball. As a first step we will show $f(A) = \Omega$. Let $y \in f(A)$, then $y = v^\alpha w_\alpha \in \mathrm{span}\,\{w_1, w_2, \ldots, w_m\}$ and $|v^\alpha w_\alpha| = |y| \le 1$. Therefore, $y \in \Omega$. Conversely, suppose $y \in \Omega$. Then $y = v^\alpha w_\alpha$ for some $v \in \mathbb{R}^m$ and $|y| = |v^\alpha w_\alpha| \le 1$. Hence, $y \in f(A)$. In a next step, we show

$$\mathcal{H}^0\left(A \cap f^{-1}(\{\cdot\})\right) = \chi_\Omega(\cdot).$$

If $y \notin \mathrm{im}\,(f) = \mathrm{span}\,\{w_1, w_2, \ldots, w_m\}$ then $f^{-1}(\{y\}) = \varnothing$ and $A \cap f^{-1}(\{y\}) = \varnothing$. Therefore, $\mathcal{H}^0\left(A \cap f^{-1}(\{y\})\right) = 0$ and $\chi_\Omega(y) = 0$ because $\Omega \subset \mathrm{im}\,(f)$.

If $y \in \mathrm{im}\,(f) \cap f(A) = \mathrm{span}\,\{w_1, w_2, \ldots, w_m\} \cap \Omega$ then the injectivity of f implies that the cardinality of $f^{-1}(\{y\}) = A \cap f^{-1}(\{y\})$ is 1. Therefore, $\mathcal{H}^0(A \cap f^{-1}(\{y\})) = 1 = \chi_\Omega(y)$. If $y \notin \Omega$ but still $y \in \mathrm{span}\,\{w_1, w_2, \ldots, w_m\}$ then $A \cap f^{-1}(\{y\}) = \varnothing$ and consequently, $\mathcal{H}^0(A \cap f^{-1}(\{y\})) = 0 = \chi_\Omega(y)$.

Using the area formula, we get

$$\begin{aligned}
&\|\sigma\|_{\bigwedge^m(\mathbb{R}^n)}\mathcal{H}^m\left(\{v\in\mathbb{R}^n \mid F(q,v^\alpha w_\alpha)\le 1\}\right)\\
&\quad=\|\sigma\|_{\bigwedge^m(\mathbb{R}^n)}\mathcal{H}^m(A)=\int_{v\in A}\|\sigma\|_{\bigwedge^m(\mathbb{R}^n)}d\mathcal{L}^m(v)\\
&\quad=\int_{v\in A}Jf(v)d\mathcal{L}^m(v)=\int_{y\in\mathbb{R}^n}\mathcal{H}^0(A\cap f^{-1}(\{y\}))d\mathcal{H}^m(y)\\
&\quad=\int_{y\in\mathbb{R}^n}\chi_\Omega(y)d\mathcal{H}^m(y)=\mathcal{H}^m\left(\overline{B_1^n(0)}\cap\operatorname{span}\{w_1,w_2,\dots,w_m\}\right)\\
&\quad=\mathcal{H}^m\left(\overline{B_1^m(0)}\right)=\varepsilon_m.
\end{aligned}$$

In the last equation we used that the n-dimensional Euclidean unit ball is centrally symmetric. This proves (3.3.4). □

Now we can compare the value of the Busemann–Hausdorff area integrand for two different Finsler metrics.

Lemma 3.3.4 ([OvdM14, Lemma 2.4, p. 286])
Suppose F_1, F_2 are two Finsler metrics on $\mathbb{R}^n$ which are both strictly positive on $\mathbb{R}^n\times(\mathbb{R}^n\setminus\{0\})$. If for $q\in\mathbb{R}^n$ there exist numbers $0<c_1(q)\le c_2(q)$ with

$$c_1(q)F_1(q,v)\le F_2(q,v)\le c_2(q)F_1(q,v)\qquad \text{for all } v\in\mathbb{R}^n,$$

then

$$m_1(q)a_m^{F_1}(q,\sigma)\le a_m^{F_2}(q,\sigma)\le c_2(q)a_m^{F_1}(q,\sigma)\qquad \text{for all } \sigma\in\mathbb{GC}_m(\mathbb{R}^n),\tag{3.3.5}$$

where $m_i(q):=c_i^m(q)$ for $i=1,2$.

Proof: We use Theorem 3.3.2 to get for $\sigma=w_1\wedge w_2\wedge\cdots\wedge w_m\in GC_m(\mathbb{R}^n)$

$$\begin{aligned}
m_1(q)a_m^{F_1}(q,w_1\wedge w_2\wedge\cdots\wedge w_m)&=c_1^m(q)\left(\frac{1}{\mathcal{H}^{m-1}(\mathbb{S}^{m-1})}\int_{\mathbb{S}^{m-1}}\frac{1}{(F_1(q,\theta^\alpha w_\alpha))^m}dS(\theta)\right)^{-1}\\
&=\left(\frac{1}{\mathcal{H}^{m-1}(\mathbb{S}^{m-1})}\int_{\mathbb{S}^{m-1}}\frac{1}{(c_1(q)F_1(q,\theta^\alpha w_\alpha))^m}dS(\theta)\right)^{-1}\\
&\le\left(\frac{1}{\mathcal{H}^{m-1}(\mathbb{S}^{m-1})}\int_{\mathbb{S}^{m-1}}\frac{1}{(F_2(q,\theta^\alpha w_\alpha))^m}dS(\theta)\right)^{-1}\\
&=a_m^{F_2}(q,w_1\wedge w_2\wedge\cdots\wedge w_m).
\end{aligned}$$

The second inequality follows similarly. □

Finally, we combine the preceding results to find the positive definiteness of a_m^F.

Theorem 3.3.5 (Positive definiteness of $\boldsymbol{a_m^F}$, [OvdM14, Corollary 2.5, p. 287])
Let F be a Finsler metric on $\mathbb{R}^n$ with

$$0 < c_1 := \inf_{\mathbb{R}^n \times \mathbb{S}^{n-1}} F(\cdot,\cdot) \leq \sup_{\mathbb{R}^n \times \mathbb{S}^{n-1}} F(\cdot,\cdot) = \|F\|_{L^\infty(\mathbb{R}^n \times \mathbb{S}^{n-1})} =: c_2 < \infty. \tag{3.3.6}$$

Then

$$m_1|\sigma| \leq a_m^F(q,\sigma) \leq m_2|\sigma| \quad \text{for all } (q,\sigma) \in \mathbb{R}^n \times \mathbb{GC}_m(\mathbb{R}^n), \tag{3.3.7}$$

where $m_i := c_i^m$ for $i = 1, 2$, that is, (D) holds true.

Proof: Through the positive homogeneity of F in its second argument, we see from (3.3.6) that

$$c_1|v| \leq |v| F\left(q, \frac{v}{|v|}\right) = F(q,v) \leq c_2|v|$$

for all $(q,v) \in \mathbb{R}^n \times \mathbb{R}^n$. For $q \in \mathbb{R}^n$ set $c_i(q) := c_i$ for $i = 1, 2$. The functions $F_1(q,v) := |v|$ and $F_2(q,v) := F(q,v)$ both are Finsler metrics on $\mathbb{R}^n$ which are strictly positive on $\mathbb{R}^n \times (\mathbb{R}^n \setminus \{0\})$. Due to Lemma 3.3.3 we find in addition that

$$a_m^{|\cdot|}(q,\sigma) = |\sigma| \quad \text{for all } (q,\sigma) \in \mathbb{R}^n \times \mathbb{GC}_m(\mathbb{R}^n),$$

where $|\cdot|$ on the left and right-hand side means the Euclidean norm on $\mathbb{R}^n$ and $\mathbb{R}^N$ respectively. Therefore, we can apply Lemma 3.3.4 to F_1 and F_2 and obtain (3.3.7) from (3.3.5). □

We conclude this chapter and the thesis with the proof of the Plateau problem in reversible Finsler space.

Proof of Theorem 3.1.1: In Sections 3.3.1, 3.3.2 and 3.3.3 we saw that the extended two-dimensional Busemann–Hausdorff area integrand $\mathfrak{a}_2^F : \mathbb{R}^n \times \mathbb{R}^N \to \mathbb{R}$ is a positive definite, semi-elliptic Cartan integrand. Recall that the Busemann–Hausdorff area functional $\text{area}_B^F(\cdot)$ in (2.1.7) is given by

$$\text{area}_\Omega^F(X) = \int_{p\in\Omega} a_m^F\left(X(p), dX_p\left(\left.\frac{\partial}{\partial u^1}\right|_p\right) \wedge \cdots \wedge dX_p\left(\left.\frac{\partial}{\partial u^m}\right|_p\right)\right) du^1 \wedge \cdots \wedge du^m.$$

Thus, $\text{area}_B^F(\cdot)$ is the Cartan functional corresponding to $I = \mathfrak{a}_2^F$. Therefore, we can apply Theorem 3.2.1 to the present situation which yields the stated result. □

Conclusion and prospects

In this thesis we solved the Plateau problem in Finsler space in arbitrary codimension for a reversible Finsler metric. We observed in Chapter 3 that the proof technique used in [OvdM14] for the codimension one case can be applied to solve the area minimisation problem in higher codimension. Essentially, only the convexity of the two-dimensional Busemann–Hausdorff area density, established in [BI12], was needed. It was especially important that the underlying Finsler metric is reversible, since otherwise all the arguments involving *symmetric* polygons in Chapter 2 do not hold.

Overath and von der Mosel, however, did not restrict themselves to reversible Finsler metrics. They considered the so-called m-harmonic symmetrisation $F_{sym}(x,y) = 2^{\frac{1}{m}} \left(F(x,y)^{-m} + F(x,-y)^{-m}\right)^{-\frac{1}{m}}$ of a Finsler metric F. Note that F_{sym} is always reversible and coincides with F, if the original Finsler metric is reversible. In [OvdM14] as a general assumption only such Finsler metrics are considered whose m-harmonic symmetrisation is also a Finsler metric. Their idea is to apply the convexity result to the reversible m-harmonic symmetrisation and use a result comparing the corresponding Finsler areas. In fact, Overath and von der Mosel showed that the Busemann–Hausdorff area functionals corresponding to F and F_{sym} coincide (see [OvdM14, Theorem 1.1 + Lemma 2.3, p. 276+285]) – but their proof is restricted to codimension one.

Further analysis is needed to see if a similar result holds true in higher codimension to fully generalise Overath and von der Mosel's work. It remains to be investigated if their results on higher regularity of solutions (see [OvdM14, Theorem 1.4, p. 281]) generalise straightforwardly to higher codimension.

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