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Higher regularity of high-codimensional disk-type surfaces minimising a Finsler area

Part I

Sven Pistre
Advisor: Heiko von der Mosel

1^{st} Geometric Analysis Festival
Organised by Hojoo Lee

February 2021

Outline

1. **Setting**
   - ➔ Finsler manifolds
   - ➔ Busemann–Hausdorff area functional $\mathscr{A}_F$
   - ➔ Plateau problem for $\mathscr{A}_F$

2. **Existence of Busemann–Hausdorff area minimisers**
   - ➔ Hildebrandt/von der Mosel framework for Cartan functionals
   - ➔ How does $\mathscr{A}_F$ fit into the Cartan theory?

Notation

Let M^m and N^n be smooth manifolds, $u\colon M \to N$ an immersion or embedding.

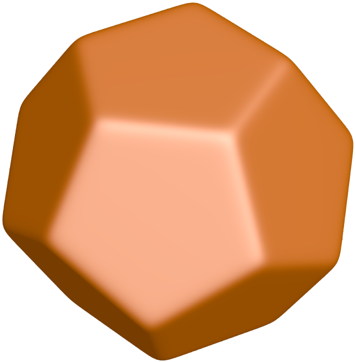
	M	N
Riem. or Finsler metrics	$\bar{g}, \overline{F}$	g, F
points	p or x	q or y
local coordinates	$(x^1, \ldots, x^m)$	

In the final results always choose

- $m = 2$,
- $M = D \subset \mathbb{R}^2$ unit disk,
- $N = \mathbb{R}^n$.

Finsler manifolds

Morally speaking:
Finsler metric = smooth family of smooth norms on tangent bundle.



This smoothened dodecahedron generates a Finsler metric on $\mathbb{R}^3$.

Finsler manifolds

A ***Finsler metric*** on N is a function $F\colon TN \to [0,\infty)$ s.t.:

(F1) *Regularity*: $F \in \mathrm{C}^k(TN \setminus \{0\}) \cap \mathrm{C}^0(TN)$, $k \in \{2, 3, \dots, +\infty\}$.

(F2) *Positive 1-homogeneity*: $F(q,tv) = tF(q,v)$ for all $t > 0$, $(q,v) \in TN$.

(F3) *Ellipticity*: For any $(q,u) \in TN \setminus \{0\}$ the "first fundamental form"

$$g^F_{q,u}(v,w) = \frac{\mathrm{d}^2}{\mathrm{d}t\mathrm{d}s}\Big|_{t,s=0} \frac{1}{2} F(q, u + tv + sw)^2$$

describes a scalar product (which depends on (q,u)!).

A Finsler metric is ***reversible*** if $F(q,v) = F(q,-v)$ for all $(q,v) \in TN$.

Busemann–Hausdorff area functional

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Suppose g is an auxiliary Riemannian metric **on the target** N.
Define the ***Riemannian volume*** of N by

Riemannian volume density

$$\mathrm{Vol}(N) = \int_N \mu_g.$$

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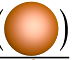
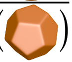
where $w_{F,g}$ is the weight function defined by

$= \Omega_m$ for all g!

$$w_{F,g}(q) = \frac{\mathcal{L}^n_{g_q}\left(\quad\right)}{\mathcal{L}^n_{g_q}\left(\quad\right)}.$$





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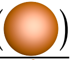

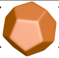

← $= \Omega_m$ for all g!

Define the ***Busemann–Hausdorff area*** of $u\colon M \to N$ by

$$\mathcal{A}_F(u) = \mathrm{Vol}_{u^\# F}(M).$$

Lemma: Vol_F (and thus $\mathcal{A}_F$) is independent of the choice of g!

Busemann–Hausdorff area functional

Lemma: $\mathcal{A}_F$ is independent of the choice of g!

Sketch of proof:

Given Riemannian metrics $g, h \in \Sigma^2(T'N)$ there is a smooth bundle isomorphism $E : TN \to TN$ such that for all vector fields $X, Y \in \mathfrak{X}(N)$

$$g(X, Y) = h(EX, EY).$$

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The induced volume densities and Lebesgue measures transform via

$$\mu_g = \det(E)\, \mu_h$$
$$\mathcal{L}^n_g = \det(E)\, \mathcal{L}^n_h.$$

Thus,

$$\int_N \frac{\Omega_n}{\mathcal{L}^n_g(B_F)}\, \mu_g = \int_N \frac{\Omega_n}{\mathcal{L}^n_h(B_F)}\, \mu_h.$$

A representation formula for $\mathcal{A}_F$

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Let $\overline{F} = u^\# F$ and $\bar{g} = u^\# g$. Then

$$\mathcal{A}_F(u) = \int_M \frac{\Omega_m}{\mathcal{L}^m_{\bar{g}_p}(B_{\overline{F}_p})} \mu_{\bar{g}}|_p.$$

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In local coordinates $(x^1, \ldots, x^m)$ around $p \in M$:

$$\frac{\Omega_m}{\mathcal{L}^m_{\bar{g}_p}(B_{\overline{F}_p})} \mu_{\bar{g}}|_p$$

$$= \frac{\Omega_m}{\mathcal{H}^m_{g_{u(p)}}(B_{F_{u(p)}} \cap \mathrm{d}u_p(T_pM))} \mu_{\bar{g}}|_p$$

$(M, \bar{g}) \xrightarrow{u} (u(M), g)$ isometric

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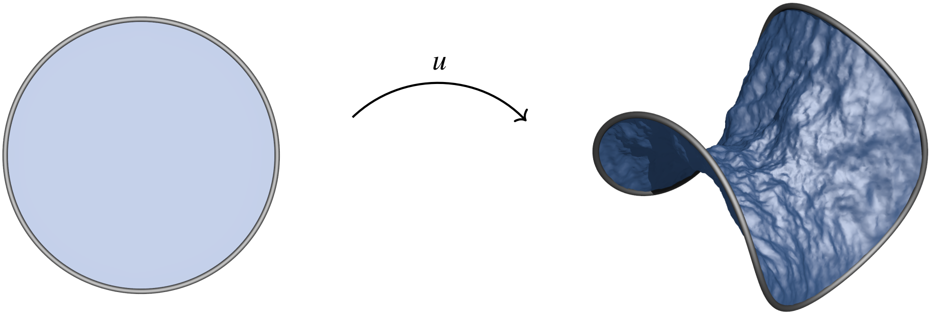
Plateau problem

Given a Jordan curve $\Gamma \subset \mathbb{R}^n$, is there a disk-type surface $u\colon D \to \mathbb{R}^n$ such that

$$\mathcal{A}_F(u) = \inf_{\mathcal{S}(\Gamma)} \mathcal{A}_F$$

in the class of competing surfaces

$$\mathcal{S}(\Gamma) = \Big\{ u \in \mathrm{W}^{1,2}(D;\mathbb{R}^n) \ :\ u|_{\partial D} \text{ parametrises } \Gamma \text{ weakly monotonically } \Big\}?$$



Existence of Busemann–Hausdorff area minimisers

Theorem (Existence) [P, von der Mosel '17]

Let $M = D$, F a Finsler and g a Riemannian metric on $N = \mathbb{R}^n$ s.t.

$$c_F|v|_{g_q} \leq F(q, v) \leq C_F|v|_{g_q} \quad \text{for all } (q, v) \in T\mathbb{R}^n$$

and suppose F satisfies a certain symmetrisation assumption †.

- Then any given rectifiable Jordan curve $\Gamma \subset \mathbb{R}^n$ bounds a *g-conformally parametrised* surface $u \in \mathcal{S}(\Gamma)$ which *minimises* $\mathcal{A}_F$ of *lower regularity*

$$C^0(\bar{D}; \mathbb{R}^n) \cap C^{0,\alpha}(D; \mathbb{R}^n) \cap W^{1,q}_{\text{loc}}(D; \mathbb{R}^n)$$

for $\alpha := (c_F/C_F)^2 \in (0, 1]$ and some $q > 2$.

† $F_{\text{sym}}(q, v) = (1/2(F(q, v)^{-2} + F(q, -v)^{-2}))^{-1/2}$ is also a Finsler metric

Definition: Cartan functionals

Suppose $u\colon M = D \to \mathbb{R}^n = N$. A ***Cartan functional*** is of the form

$$\mathcal{E}(u) = \int_D E\Big(u, \tfrac{\partial u}{\partial x^1} \wedge \tfrac{\partial u}{\partial x^2}\Big)\, \mathrm{d}x^1 \mathrm{d}x^2$$

where the ***Cartan density*** $E \in \mathrm{C}^0(\mathbb{R}^n \times \bigwedge^2 \mathbb{R}^n)$ is

- positively 1-homogeneous,
- convex
- and of linear growth: $m_1|\sigma|_{\bigwedge^2\mathbb{R}^n} \leq E(q,\sigma) \leq m_2|\sigma|_{\bigwedge^2\mathbb{R}^n}$

in the second argument.

Remarks: Cartan functionals

$$\mathcal{E}(u) = \int_D E\left(u, \frac{\partial u}{\partial x^1} \wedge \frac{\partial u}{\partial x^2}\right) \mathrm{d}x^1 \mathrm{d}x^2$$

- Positive 1-homogeneity of E $\Rightarrow$ (orient.-pres.) diffeomorphism-invariance of $\mathcal{E}$
  - ➔ Conversely:
    $\mathcal{E}(u) = \int_D e(u, \mathrm{d}u)\,\mu_g$ is diffeomorphism-invariant $\Rightarrow$ existence of a Cartan density E

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- Convexity and linear growth of E $\quad\Rightarrow\quad$ weak lower semicontinuity of $\mathcal{E}$ in $\mathrm{W}^{1,2}$
  - → Indeed:
    E convex $\quad\Rightarrow\quad$ $e(u, \mathrm{d}u) = E\Big(u, \tfrac{\partial u}{\partial x^1} \wedge \tfrac{\partial u}{\partial x^2}\Big)$ polyconvex (and thus, quasiconvex)

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- The three above conditions on E guarantee existence of $\mathcal{E}$-minimisers.
- Further conditions yield higher interior regularity and also regularity at the boundary.
- Cartan functionals were investigated by Hildebrandt and von der Mosel ['99-'09].

$\mathcal{A}_F$ as a Cartan functional

For some auxiliary Riemannian metric g **on the target** N, consider the function $A_{F,g} : \bigwedge_s^m TN \to \mathbb{R}$ defined by

This is the Euclidean area integrand

$$A_{F,g}(q,\sigma) = \frac{\Omega_m}{\mathcal{H}^m_{g_q}(B_{F_q} \cap \langle \sigma \rangle)} |\sigma|_{g_q, \bigwedge^m T_q N}$$

where $\langle \sigma \rangle = \{v \in T_q N \; : \; \sigma \wedge v = 0\}$ is the m-dimensional subspace of $T_q N$ spanned by σ.

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By the representation formula for $\mathcal{A}_F$ in local coordinates at p,

$$\mathcal{A}_F(u) = \int_M A_{F,g}\Big(u(p), \frac{\partial u}{\partial x^1}\Big|_p \wedge \ldots \wedge \frac{\partial u}{\partial x^m}\Big|_p\Big)\, \mathrm{d}x^1 \ldots \mathrm{d}x^m.$$

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The integrand $A_{F,g} \colon \bigwedge_s^m TN \to \mathbb{R}$ is

- absolutely 1-homogeneous
- convexly extendible to $\bigwedge^m TN$
  - → for $n = m + 1$ due to [Busemann '49]
  - → for $m = 2$, $n \in \mathbb{N}$ due to [Burago, Ivanov '12]

in the second argument.

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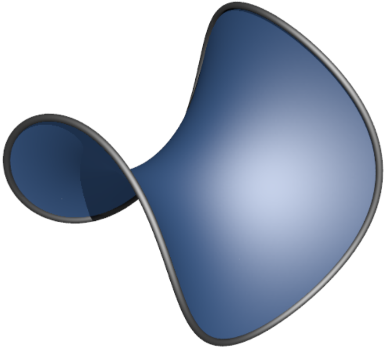
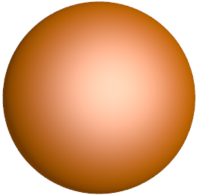
Some images



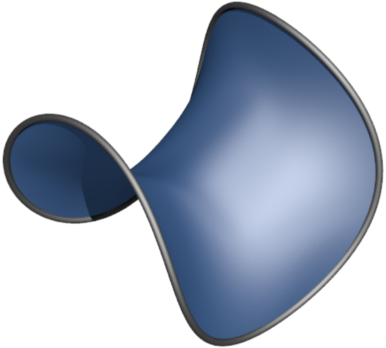
Minimising surface



Finsler unit ball

Numerical framework due to Henrik Schumacher
Created with WOLFRAM MATHEMATICA 11
Rendered with POV-Ray 3.7

Some images



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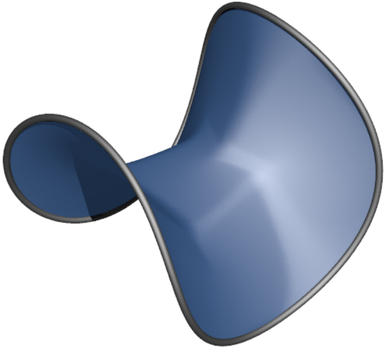

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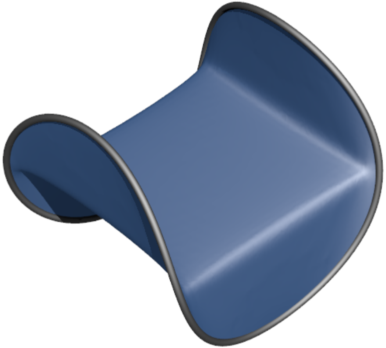

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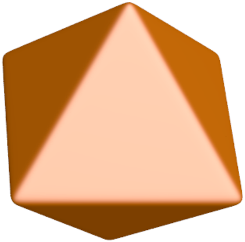

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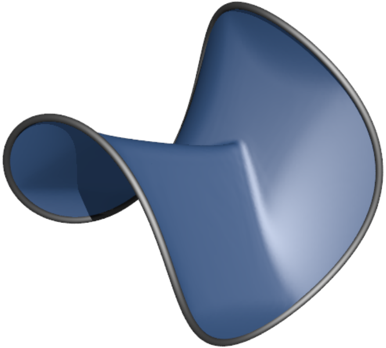

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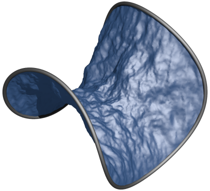
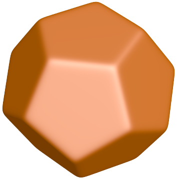
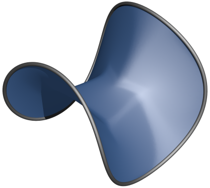
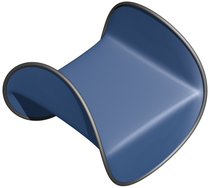
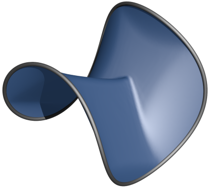




See you in part II of the lecture series!







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Higher regularity of high-codimensional disk-type surfaces minimising a Finsler area

Part II

Sven Pistre
Advisor: Heiko von der Mosel

1^{st} Geometric Analysis Festival
Organised by Hojoo Lee

February 2021

Outline

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1. **Higher regularity via the framework for Cartan functionals**

2. **Ingredient: Perfect dominance functions**

3. **Ingredient: Radon transform**

Higher regularity via the framework for Cartan functionals

Theorem [Overath, von der Mosel '13]

In codimension 1:

Let $M = D$, F a Finsler metric on $N = \mathbb{R}^3$ and $g = \langle \cdot, \cdot \rangle_{\mathbb{R}^3}$.
There is a constant $c > 0$ s.t.

- If

$$\sup_{q \in \mathbb{R}^3} [\![F(q, \cdot) - |\cdot|_{\mathbb{R}^3}]\!]_{C^2(\mathbb{S}^2)} < c,$$

  then every conformal minimiser u of $\mathcal{A}_F$ in $\mathcal{S}(\Gamma)$ is of class

$$W^{2,2}_{\text{loc}}(D; \mathbb{R}^3) \cap C^{1,\mu}(D; \mathbb{R}^3)$$

  for some $\mu \in (0, 1)$.

Theorem [P '21+ε]

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In arbitrary codimension:

Let $M = D$, F a Finsler metric on $N = \mathbb{R}^n$ and g a Riemannian metric.
There is a constant $c = c(g, N) > 0$ s.t.

- If

$$\sup_{q \in \mathbb{R}^n} [\![F(q, \cdot) - |\cdot|_{g_q}]\!]_{C^2(\mathbb{S}^{n-1}_{g_q})} < c,$$

then every g-conformal minimiser u of $\mathcal{A}_F$ in $\mathcal{S}(\Gamma)$ is of class

$$W^{2,2}_{\text{loc}}(D; \mathbb{R}^n) \cap C^{1,\mu}(D; \mathbb{R}^n)$$

for some $\mu \in (0, 1)$.

Theorem ?

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Theorem [P '21+ε]

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Ingredient: Perfect dominance functions

Theorem [Hildebrandt, von der Mosel '03]

Suppose $\mathcal{E}$ is a Cartan functional with Cartan density E,
i.e. $E \in \mathrm{C}^0(\mathbb{R}^n \times \bigwedge^2 \mathbb{R}^n)$ is

- positively 1-homogeneous,
- convex,
- and of linear growth.

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Then every conformal minimiser u of $\mathcal{E}$ in $\mathcal{S}(\Gamma)$ is of class

$$\mathrm{W}^{2,2}_{\text{loc}}(D;\mathbb{R}^n) \cap \mathrm{C}^{1,\mu}(D;\mathbb{R}^n)$$

for some $\mu \in (0,1)$, if the Cartan density E

- is of class $\mathrm{C}^2(\mathbb{R}^n \times (\bigwedge^2 \mathbb{R}^n \setminus \{0\}))$
- **and it possesses a *perfect* dominance function.**

Definition: Dominance functions

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A function $D \in \mathrm{C}^0(\mathbb{R}^n \times \mathbb{R}^{n\times 2})$ is a ***dominance function***
for an Cartan density $E \in \mathrm{C}^0(\mathbb{R}^n \times \bigwedge^2\mathbb{R}^n)$ if

- $E(q, P_1 \wedge P_2) \leq D(q, P)$ for all $q \in \mathbb{R}^n$ and $P = (P_1, P_2) \in \mathbb{R}^{n\times 2}$,
- $E(q, P_1 \wedge P_2) = D(q, (P_1, P_2))$ if and only if $|P_1|^2_{\mathbb{R}^n} = |P_2|^2_{\mathbb{R}^n}$ and $\langle P_1, P_2\rangle_{\mathbb{R}^n} = 0$,

and in the second argument the dominance function D is

- positively 2-homogeneous,
- and of quadratic growth: $\mu_1|P|^2_{\mathbb{R}^{n\times 2}} \leq D(q, P) \leq \mu_2|P|^2_{\mathbb{R}^{n\times 2}}$.

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A function $D \in C^0(\mathbb{R}^n \times \mathbb{R}^{n\times 2})$ is a ***dominance function***
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A dominance function D is called ***perfect*** if

- $D \in C^2(\mathbb{R}^n \times (\mathbb{R}^{n\times 2} \setminus \{0\}))$ and
- for $R > 0$ there is $\lambda(R) > 0$ s.t. for all $|q|_{\mathbb{R}^n} < R$ the function

$$P \mapsto D(q, P) - \lambda(R)\tfrac{1}{2}|P|^2_{\mathbb{R}^{n\times 2}}$$

  is convex (i.e. $D(q, \cdot)$ is $\lambda(R)$-convex).

Perfect dominance functions: Existence

Theorem [Hildebrandt, von der Mosel '03]

Suppose $E \in C^2(\mathbb{R}^n \times (\bigwedge^2\mathbb{R}^n \setminus \{0\}))$ is positively 1-homogeneous, of linear growth (with constants m_1 and m_2) and ***uniformly elliptic***,
i.e. there is $\lambda > 0$ s.t. for all $q \in \mathbb{R}^n$ the function

$$\sigma \mapsto E(q,\sigma) - \lambda|\sigma|_{\bigwedge^2\mathbb{R}^n}$$

is convex.
Then for every $k > k_0(m_1, m_2, \lambda)$ † the new integrand

$$(q,\sigma) \mapsto E(q,\sigma) + k|\sigma|_{\bigwedge^2\mathbb{R}^n}$$

possesses a perfect dominance function.

This is the Euclidean area integrand

† $k_0(m_1, m_2, \lambda) = 2m_2 - \min(2\lambda, m_1)$

Perfect dominance functions: Existence

Corollary [Overath, von der Mosel '13]

Suppose E is as before and

This is the Euclidean area integrand

$$\sup_{q\in\mathbb{R}^n} [\![E(q,\cdot) - |\cdot|_{\bigwedge^2\mathbb{R}^n}]\!]_{C^2\left(\mathbb{S}^{\binom{n}{2}-1}\right)} < \frac{1}{5}$$

then E itself possesses a perfect dominance function.

Ingredient: Radon transform

Goal

For $m = 2$ and $N = \mathbb{R}^n$:
Show that the integrand $A_{F,g}\colon \bigwedge_s^2 T\mathbb{R}^n \to \mathbb{R}$ of $\mathcal{A}_F$ satisfies

(i) $A_{F,g} \in C^2(\bigwedge_s^2 T\mathbb{R}^n \setminus \{0\})$,

(ii) $A_{F,g}$ is uniformly elliptic

(iii) and $\sup_{q\in\mathbb{R}^n} [\![A_{F,g}(q,\cdot) - |\cdot|_{g_q, \bigwedge^2 T_q\mathbb{R}^n}]\!]_{C^2\left(\mathbb{S}_{g_q}^{\binom{n}{2}-1}\right)} < \frac{1}{5}$.

Compare condition (iii) to the smallness condition on

$$\sup_{q\in\mathbb{R}^n} [\![F(q,\cdot) - |\cdot|_{g_q}]\!]_{C^2(\mathbb{S}_{g_q}^{n-1})}$$

in the main theorem.

These three goals can be achieved by using functional analytic properties of the so-called ***Radon transform***.

Motivation: An easy calculation

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Recall from earlier for $\sigma \in \bigwedge_s^m T_q N$:

$$A_{F,g}(q,\sigma) = \frac{\Omega_m}{\mathcal{H}^m_{g_q}\left(B_{F_q} \cap \langle \sigma \rangle\right)} |\sigma|_{g_q, \bigwedge^m T_q N}.$$

Using spherical coordinates in the m-dim. Hilbert space $(\langle \sigma \rangle, g_q|_{\langle \sigma \rangle})$:

$$\mathcal{H}^m_{g_q}(B_{F_q} \cap \langle \sigma \rangle) = \int_{\mathbb{S}^{m-1}_{g_q|_{\langle \sigma \rangle}}} \int_0^\infty \chi_{B_{F_q} \cap \langle \sigma \rangle}(s\theta) \ \ s^{m-1} \mathrm{d}s \ \mathrm{d}\mathcal{H}^{m-1}_{g_q}(\theta).$$

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$$\frac{\mathcal{H}^m_{g_q}\big(B_{F_q} \cap \langle\sigma\rangle\big)}{\Omega_m} = \frac{1}{m\Omega_m} \int_{\mathbb{S}^{m-1}_{g_q|_{\langle\sigma\rangle}}} F(q,\cdot)^{-m} \,\mathrm{d}\mathcal{H}^{m-1}_{g_q}$$

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$(\langle\sigma\rangle, g_q|_{\langle\sigma\rangle}) \hookrightarrow (T_q M, g_q)$ isometry

$$= \frac{1}{\omega_{m-1}} \int_{\mathbb{S}^{n-1}_{g_q} \cap \langle\sigma\rangle} F(q,\cdot)^{-m} \, \mathrm{d}\mathcal{H}^{m-1}_{g_q}.$$

The Radon transform: Definition

Let Gr^n_m denote the Grassmannian, the set of m-dimensional subspaces of $\mathbb{R}^n$.

The linear operator $\mathcal{R}\colon \mathrm{C}^0(\mathbb{S}^{n-1}) \to \mathrm{C}^0(\mathrm{Gr}^n_m)$ defined by

$$\mathcal{R}[f](E) = \frac{1}{\omega_{m-1}} \int_{\mathbb{S}^{n-1}\cap E} f \, \mathrm{d}\mathcal{H}^{m-1}$$

is called the ***Radon transform*** on $\mathbb{R}^n$.
This construction can be lifted to a linear operator $\mathcal{R}_g$ on the sphere bundle and the Grassmann bundle of (N, g).

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This construction can be lifted to a linear operator $\mathcal{R}_g$ on the sphere bundle and the Grassmann bundle of (N, g).

Corollary

The area integrand $A_{F,g}$ satisfies

$$A_{F,g}(q,\sigma) = \frac{1}{\mathcal{R}_g[F(q,\cdot)^{-m}](\langle\sigma\rangle)} |\sigma|_{g_q, \wedge^m T_q N}.$$

The Radon transform: Properties

- Downside to using $A_{F,g}$:
  In high codimension the base space $\bigwedge_s^m TN$ is a bundle of cones and difficult to handle.

- Benefits of the Radon transform:
  Base spaces $\mathbb{S}^{n-1}$ and Gr_m^n are homogeneous $\mathrm{O}(n)$-spaces with a lot of algebraic and geometric structure.

- By the Plücker embedding both base spaces are related via

$$\mathrm{Gr}_m^n \cong \left(\mathbb{S}\left(\bigwedge^m \mathbb{R}^n\right) \cap \bigwedge_s^m \mathbb{R}^n \right) \Big/ \mathbb{Z}_2.$$

The Radon transform: Properties

For $E \in \mathrm{Gr}^n_m$: $\mathcal{R}[f](E) = \frac{1}{\omega_{m-1}} \int_{\mathbb{S}^{n-1} \cap E} f \, \mathrm{d}\mathcal{H}^{m-1}$.

Lemma ($\mathrm{O}(n)$-Equivariance)

For any $f \in \mathrm{C}^0(\mathbb{S}^{n-1})$ and $Q \in \mathrm{O}(n)$:

$$\mathcal{R}[f] \circ Q = \mathcal{R}[f \circ Q].$$

The Radon transform: Properties

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Lemma (O(n)-Equivariance)
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$$\mathcal{R}[f] \circ Q = \mathcal{R}[f \circ Q].$$

Taking derivative at $Q = \mathrm{Id}_{\mathbb{R}^n}$ leads to:

Theorem (Differentiability)
Suppose $f \in \mathrm{C}^1(\mathbb{S}^{n-1})$. Then $\mathcal{R}[f] \in \mathrm{C}^1(\mathrm{Gr}_m(\mathbb{R}^n))$ and for all $X \in \mathfrak{o}(n)$:

$$\mathrm{d}(\mathcal{R}[f])(\mathcal{K}_X) = \mathcal{R}[\mathrm{d}f(\mathcal{K}_X)].$$

↑ ↑ Fundamental vector fields generated by O(n)-actions

The Radon transform: Properties

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Theorem (Higher differentiability)

For any $k \geq 0$ the Radon transform

$$\mathcal{R} \colon C^k(\mathbb{S}^{n-1}) \to C^k(\mathrm{Gr}^n_m)$$

is a continuous operator with explicit bounds for its operator norm.

The Radon transform: Properties

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Theorem (Invertibility, [Helgason, '90])

The Radon transform (restricted to *even* functions on the sphere) is an invertible† operator.

---

†Helgason gives an explicit inversion formula

The Radon transform: Summary

- We needed to prove three properties for the area integrand $A_{F,g}$ to apply the theorems of Hildebrandt/von der Mosel:
  (i) $A_{F,g} \in C^2(\wedge_s^2 T\mathbb{R}^n \setminus \{0\})$,
  → Use higher differentiability of $\mathcal{R}$
  (ii) $A_{F,g}$ is uniformly elliptic
  → Use invertibility of $\mathcal{R}$
  (iii) and $\sup_{q\in\mathbb{R}^n} [\![A_{F,g}(q,\cdot) - |\cdot|_{g_q,\wedge^2 T_q\mathbb{R}^n}]\!]_{C^2\left(\mathbb{S}_{g_q}^{\binom{n}{2}-1}\right)} < \frac{1}{5}$.
  → Use operator norm bounds for $\mathcal{R}$

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- We needed to prove three properties for the area integrand $A_{F,g}$ to apply the theorems of Hildebrandt/von der Mosel:
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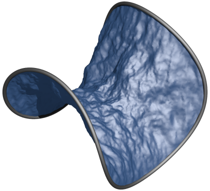
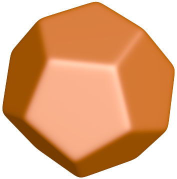
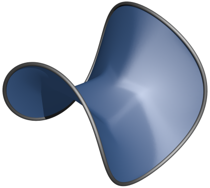
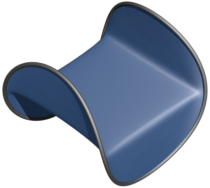
  (iii) and $\sup_{q\in\mathbb{R}^n} [\![A_{F,g}(q,\cdot) - |\cdot|_{g_q,\Lambda^2 T_q\mathbb{R}^n}]\!]_{C^2\left(\mathbb{S}_{g_q}^{\binom{n}{2}-1}\right)} < \frac{1}{5}$.
    → Use operator norm bounds for $\mathcal{R}$
- The main difficulty in high codimension is that $\Lambda_s^m TN$ is not a **vector** bundle!
- Use the Radon transform and exploit the nice algebraic structure of the base spaces!

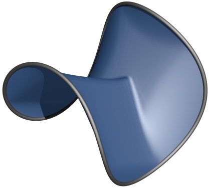




Thank you for your attention!







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