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Mathematics
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The Radon transform and higher regularity of surfaces minimising a Finsler area

Joint work with Heiko von der Mosel

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1. **Setting**
   - ➔ Finsler manifolds
   - ➔ Busemann–Hausdorff area functional $\mathscr{A}_F$
   - ➔ Plateau problem for $\mathscr{A}_F$

2. **Hildebrandt/von der Mosel framework for Cartan functionals**
   - ➔ How does $\mathscr{A}_F$ fit into the Cartan theory?
   - ➔ Existence of area minimisers

3. **Higher regularity**
   - ➔ Ingredient: Perfect dominance functions
   - ➔ Ingredient: Radon transform

Setting

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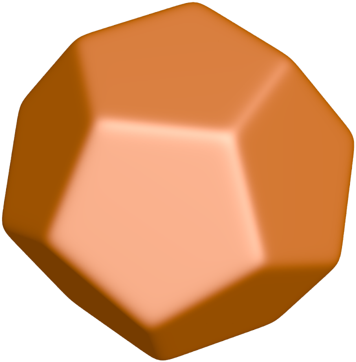
Let M^m and N^n be smooth manifolds, $f\colon M \to N$ an immersion or embedding. Denote local coordinates on M by $(u^1, \ldots, u^m)$.
Later:

- $m = 2$
- $M = D \subset \mathbb{R}^2$ unit disk
- $N = \mathbb{R}^n$

Finsler manifolds

Morally speaking:
Finsler metric = smooth family of smooth norms on tangent bundle.



This smoothened dodecahedron generates a Finsler metric on $\mathbb{R}^3$.

A ***Finsler metric*** on N is a function $F\colon TN \to [0,\infty)$ s.t.:

(F1) *Regularity*: $F \in C^k(TN \setminus \{0\}) \cap C(TN)$, $k \in \{2,3,\dots,+\infty\}$.

(F2) *Positive 1-homogeneity*: $F(x,tv) = tF(x,v)$ for all $t > 0$, $(x,v) \in TN$.

(F3) *Ellipticity*: For any $(x,v) \in TN \setminus \{0\}$ the "first fundamental form"

$$g^F_{x,v}(u,w) = \frac{d^2}{dtds}\Big|_{t,s=0} \frac{1}{2} F(x, v + tu + sw)^2$$

describes a scalar product (which depends on (x,v)!).

A Finsler metric is ***reversible*** if $F(x,v) = F(x,-v)$ for all $(x,v) \in TN$.

Busemann–Hausdorff area functional

Suppose $\bar{g}$ is an auxiliary Riemannian metric **on the domain** M. Let $\bar{F} = f^{\#}F$ be the pull-back Finsler metric on M.

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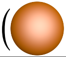
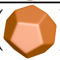
$$\mathcal{A}_F(f) = \int_M w_{\bar{F},\bar{g}}\mu_{\bar{g}}$$

where $\mu_{\bar{g}}$ is the Riemannian volume density of $\bar{g}$ and $w_{\bar{F},\bar{g}}$ is the weight function defined by

$$w_{\bar{F},\bar{g}}(p) = \frac{\mathcal{H}^m_{\bar{g}_p}(B_{\bar{g}_p})}{\mathcal{H}^m_{\bar{g}_p}(B_{\bar{F}_p})} = \frac{\mathrm{Vol}_{\bar{g}}(\cdot)}{\mathrm{Vol}_{\bar{g}}(\cdot)}.$$





$= \Omega_m$ for all $\bar{g}$!

Here, $B_{\bar{F}_p}$ and $B_{\bar{g}_p}$ are the $\bar{F}$- and $\bar{g}$-unit balls in T_pM and $\mathcal{H}^m_{\bar{g}_p}$ is the Hausdorff measure in the Hilbert space $(T_pM, \bar{g}_p)$.

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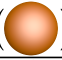
$\mathrm{Vol}_{\bar{g}}($$)$ / $\mathrm{Vol}_{\bar{g}}($$)$.

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Lemma: $\mathcal{A}_F$ is independent of the choice of $\bar{g}$!

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Sketch of proof:

Given Riemannian metrics $\bar{g}, \bar{h} \in \Sigma^2(T'M)$ there is a smooth bundle isomorphism $B\colon TM \to TM$ such that for all vector fields $X, Y \in \mathfrak{X}(M)$

$$\bar{g}(X, Y) = \bar{h}(BX, Y).$$

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The induced volume densities and Hausdorff measures transform as

$$\mu_{\bar{g}} = \sqrt{\det(B)}\,\mu_{\bar{h}}$$
$$\mathcal{H}^m_{\bar{g}} = \sqrt{\det(B)}\,\mathcal{H}^m_{\bar{h}}$$

Thus,

$$\int_M \frac{\Omega_m}{\mathcal{H}^m_{\bar{g}}(B_{\bar{F}})}\mu_{\bar{g}} = \int_M \frac{\Omega_m}{\mathcal{H}^m_{\bar{h}}(B_{\bar{F}})}\mu_{\bar{h}}.$$

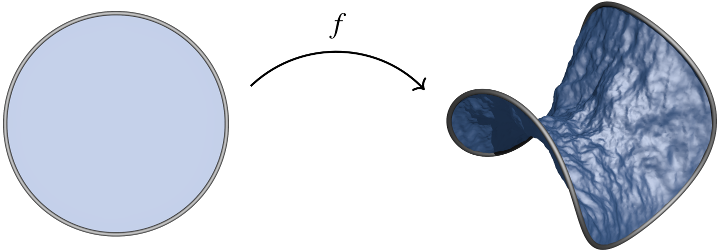
Plateau problem

Given a Jordan curve $\Gamma \subset \mathbb{R}^n$, is there a disk-type surface $f\colon D \to \mathbb{R}^n$ such that

$$\mathcal{A}_F(f) = \inf_{\mathcal{S}(\Gamma)} \mathcal{A}_F$$

in the class of competing surfaces

$$\mathcal{S}(\Gamma) = \{f \in W^{1,2}(D;\mathbb{R}^n) \ : \ f|_{\partial D} \text{ parametrises } \Gamma \text{ weakly monotonically}\}?$$



Hildebrandt/von der Mosel framework for Cartan functionals

Definition: Cartan functionals

If $f\colon M = D \to \mathbb{R}^n = N$ then a ***Cartan functional*** is of the form

$$\mathcal{C}(f) = \int_D C\Big(f, \tfrac{\partial f}{\partial u^1} \wedge \tfrac{\partial f}{\partial u^2}\Big)\, \mathrm{d}u^1 \mathrm{d}u^2$$

where the ***parametric integrand*** $C \in C(\mathbb{R}^n \times \mathbb{R}^{\binom{n}{2}})$ is

- positively 1-homogeneous,
- convex
- and of linear growth: $m_1 |\sigma|_{\mathbb{R}^{\binom{n}{2}}} \leq C(x,\sigma) \leq m_2 |\sigma|_{\mathbb{R}^{\binom{n}{2}}}$

in the second argument.

Remarks: Cartan functionals

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- Positive 1-homogeneity of C corresponds to diffeomorphism-invariance of $\mathcal{C}$.
- Convexity and linear growth of C correspond to lower semicontinuity of $\mathcal{C}$.
- The three above conditions on C guarantee existence of $\mathcal{C}$-minimisers.

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- The three above conditions on C guarantee existence of $\mathcal{C}$-minimisers.
- Further conditions yield higher interior regularity (more on that later) and also regularity at the boundary.
- A lot of results Cartan functionals were investigated by [Hildebrandt, vdM '99-'09].

$\mathcal{A}_F$ as a Cartan functional

For some auxiliary Riemannian metric g **on the target** N, consider the function $A_{F,g} : \bigwedge_s^m TN \to \mathbb{R}$ defined by

This is the Euclidean area integrand

$$A_{F,g}(x, \sigma) = \frac{\Omega_m}{\mathcal{H}^m_{g_x}(B_{F_x} \cap \langle \sigma \rangle)} |\sigma|_{g_x, \bigwedge^m T_x N}$$

where $\langle \sigma \rangle = \{v \in T_x N \ : \ \sigma \wedge v = 0\}$ is the m-dimensional subspace of $T_x N$ spanned by σ.

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Recall that $\bar{F} = f^\# F$ and choose $\bar{g} = f^\# g$ in the definition of $\mathcal{A}_F$ to see that

$$\mathcal{A}_F(f) = \int_M \frac{\Omega_m}{\mathcal{H}^m_{\bar{g}_p}(B_{\bar{F}_p})} \mu_{\bar{g}}|_p$$

$f : (M, f^\# g) \to (f(M), g)$ isometric

$$= \int_M \frac{\Omega_m}{\mathcal{H}^m_{g_{f(p)}}(B_{F_{f(p)}} \cap df_p(T_p M))} \mu_{\bar{g}}|_p$$

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$$= \int_M A_{F,g}\Big(f(p), \frac{\partial f}{\partial u^1}\Big|_p \wedge \dots \wedge \frac{\partial f}{\partial u^m}\Big|_p\Big)\, \mathrm{d}u^1 \dots \mathrm{d}u^m.$$

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Thus in local coordinates $p = (u^1, \ldots, u^m)$,

$$\mathcal{A}_F(f) = \int_M A_{F,g}\Big(f, \tfrac{\partial f}{\partial u^1} \wedge \ldots \wedge \tfrac{\partial f}{\partial u^m}\Big)\, \mathrm{d}u^1 \ldots \mathrm{d}u^m.$$

The integrand $A_{F,g}\colon\ \bigwedge_s^m TN \to \mathbb{R}$ is

- absolutely 1-homogeneous
- convexly extendible to $\bigwedge^m TN$
  - ➔ for $n = m + 1$ due to [Busemann '49]
  - ➔ for $m = 2,\ n \in \mathbb{N}$ due to [Burago, Ivanov '12]

in the second argument.

Theorem (Existence) [P, vdM ’17]

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Let $M = D$, F a Finsler and g a Riemannian metric on $N = \mathbb{R}^n$ s.t.

$$c_F |v|_{g_x} \leq F(x, v) \leq C_F |v|_{g_x} \quad \text{for all } (x, v) \in T\mathbb{R}^n$$

and suppose F satisfies a certain symmetrisation assumption†.

- Then any given rectifiable Jordan curve $\Gamma \subset \mathbb{R}^n$ bounds a surface $f \in \mathcal{S}(\Gamma)$ which minimises $\mathcal{A}_F$.

† $F_{\text{sym}}(x, v) = (1/2(F(x, v)^{-m} + F(x, -v)^{-m}))^{-1/m}$ is also a Finsler metric

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- In addition, the minimiser f is g-conformally parametrised on D:

$$f^\# g(\cdot\,,\cdot) = e^{2\lambda} \langle \cdot\,,\cdot \rangle_{\mathbb{R}^2} \quad \mathcal{H}^2\text{– a.e. on } D.$$

for some smooth function λ.

$^\dagger F_{\mathrm{sym}}(x, v) = (1/2(F(x, v)^{-m} + F(x, -v)^{-m}))^{-1/m}$ is also a Finsler metric

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  for some smooth function λ.
- Furthermore, the minimiser f is of class

  $$C(\overline{D};\mathbb{R}^n) \cap C^{0,\alpha}(D;\mathbb{R}^n) \cap W^{1,q}_{\mathrm{loc}}(D;\mathbb{R}^n)$$

  for $\alpha := (c_F/C_F)^2 \in (0,1]$ and some $q > 2$.

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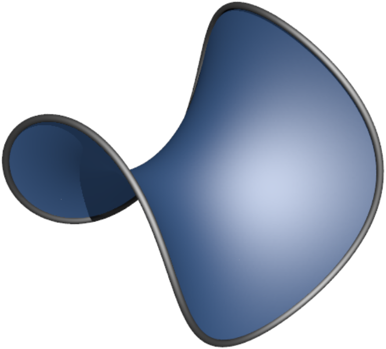
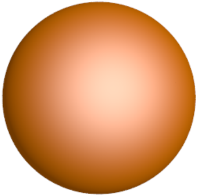
Some images



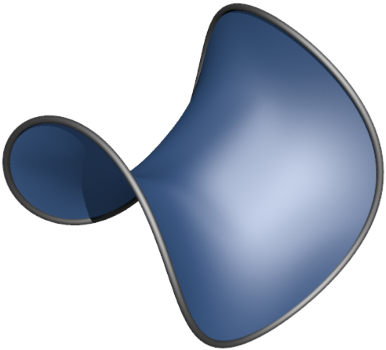
Minimising surface



Finsler unit ball

Numerical calculation due to Henrik Schumacher

Some images



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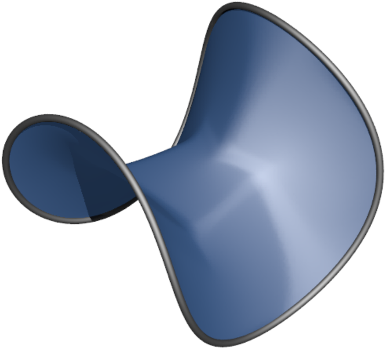

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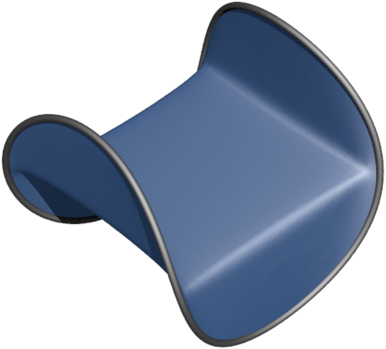

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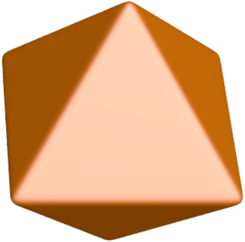

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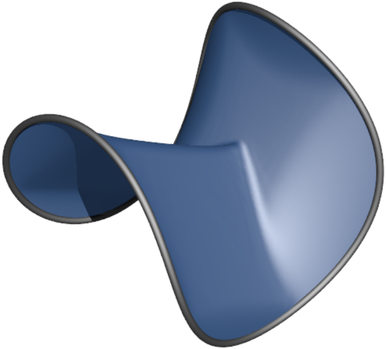

Minimising surface

Finsler unit ball

Higher regularity

Theorem [Overath, vdM ’13]

In codimension 1:

Let $M = D$, F a Finsler metric on $N = \mathbb{R}^3$ and $g = \langle \cdot, \cdot \rangle_{\mathbb{R}^3}$.
There is a constant $c > 0$ s.t.

- If

$$\sup_{x \in \mathbb{R}^3} \left\| F(x, \cdot) - |\cdot|_{\mathbb{R}^3} \right\|_{C^2(\mathbb{S}^2)} < c,$$

  then every conformal minimiser f of $\mathcal{A}_F$ in $\mathcal{S}(\Gamma)$ is of class

$$W^{2,2}_{\mathrm{loc}}(D; \mathbb{R}^3) \cap C^{1,\mu}(D; \mathbb{R}^3)$$

  for some $\mu \in (0, 1)$.

Theorem [P '20+ε]

In arbitrary codimension:

Let $M = D$, F a Finsler metric on $N = \mathbb{R}^n$ and g a Riemannian metric. There is a constant $c = c(g, n) > 0$ s.t.

- If

$$\sup_{x\in\mathbb{R}^n} \left\| F(x,\cdot) - |\cdot|_{g_x} \right\|_{C^2(\mathbb{S}^{n-1}_{g_x})} < c,$$

  then every g-conformal minimiser f of $\mathcal{A}_F$ in $\mathcal{S}(\Gamma)$ is of class

$$W^{2,2}_{\mathrm{loc}}(D;\mathbb{R}^n) \cap C^{1,\mu}(D;\mathbb{R}^n)$$

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Theorem ?

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$$\|F - g\|_{C^2(S(TN))} < c,$$

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Higher regularity

Ingredient: Perfect dominance functions

Theorem [Hildebrandt, vdM ’03]

Suppose $\mathcal{C}$ is a Cartan functional with parametric integrand C, i.e. $C \in C(\mathbb{R}^n \times \mathbb{R}^{\binom{n}{2}})$ is

- positively 1-homogeneous,
- convex,
- and of linear growth.

Then every conformal minimiser f of $\mathcal{C}$ in $\mathcal{S}(\Gamma)$ is of class

$$W^{2,2}_{\mathrm{loc}}(D;\mathbb{R}^n) \cap C^{1,\mu}(D;\mathbb{R}^n)$$

for some $\mu \in (0,1)$, if the parametric integrand C further

- is of class $C^2(\mathbb{R}^n \times (\mathbb{R}^{\binom{n}{2}} \setminus \{0\}))$
- **and it possesses a *perfect* dominance function.**

Definition: Dominance functions

A function $D \in C(\mathbb{R}^n \times \mathbb{R}^{n\times 2})$ is a ***dominance function*** for a parametric integrand $C \in C(\mathbb{R}^n \times \mathbb{R}^{\binom{n}{2}})$ if

- $C(x, p \wedge q) \leq D(x, P)$ for all $x \in \mathbb{R}^n$ and $P = (p, q) \in \mathbb{R}^{n\times 2}$,
- $C(x, p \wedge q) = D(x, (p, q))$ if and only if $|p|^2_{\mathbb{R}^n} = |q|^2_{\mathbb{R}^n}$ and $\langle p, q\rangle_{\mathbb{R}^n} = 0$,

and in the second argument the dominance function D is

- positively 2-homogeneous,
- and of quadratic growth: $\mu_1 |P|^2_{\mathbb{R}^{n\times 2}} \leq D(x, P) \leq \mu_2 |P|^2_{\mathbb{R}^{n\times 2}}$.

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A dominance function D is called ***perfect*** if

- $D \in C^2(\mathbb{R}^n \times (\mathbb{R}^{n\times 2} \setminus \{0\}))$ and
- for $R > 0$ there is $\lambda(R) > 0$ s.t. for all $|x|_{\mathbb{R}^n} < R$ the function

$$P \mapsto D(x, P) - \lambda(R)\tfrac{1}{2}|P|^2_{\mathbb{R}^{n\times 2}}$$

  is convex.

Existence of perfect dominance functions

Theorem [Hildebrandt, vdM '03]

Suppose $C \in C^2(\mathbb{R}^n \times (\mathbb{R}^{\binom{n}{2}} \setminus \{0\}))$ is positively 1-homogeneous, of linear growth (with constants m_1 and m_2) and ***uniformly elliptic***, i.e. there is $\lambda > 0$ s.t. for all $x \in \mathbb{R}^n$ the function

$$\sigma \mapsto C(x,\sigma) - \lambda|\sigma|_{\mathbb{R}^{\binom{n}{2}}}$$

is convex.

Then for every $k > k_0(m_1, m_2, \lambda)^{\dagger}$ the new integrand

$$(x,\sigma) \mapsto C(x,\sigma) + k|\sigma|_{\mathbb{R}^{\binom{n}{2}}}$$

possesses a perfect dominance function.

This is the Euclidean area integrand

$^{\dagger} k_0(m_1, m_2, \lambda) = 2(m_2 - \min(\lambda, m_1/2))$

Existence of perfect dominance functions

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is convex.

Then for every $k > k_0(m_1, m_2, \lambda)$ the new integrand

$$(x,\sigma) \mapsto C(x,\sigma) + k|\sigma|_{\mathbb{R}^{\binom{n}{2}}}$$

possesses a perfect dominance function.

This is the Euclidean area integrand

Corollary [Overath, vdM '13]

Suppose C is as above and

$$\sup_{x\in\mathbb{R}^n} \left\| C(x,\cdot) - |\cdot|_{\mathbb{R}^{\binom{n}{2}}} \right\|_{C^2\left(\mathbb{S}^{\binom{n}{2}-1}\right)} < \frac{1}{5}$$

then C itself possesses a perfect dominance function.

Higher regularity

Ingredient: Radon transform

Goal

For $m = 2$ and $N = \mathbb{R}^n$:
Show that the integrand $A_{F,g}\colon \bigwedge_s^2 T\mathbb{R}^n \to \mathbb{R}$ of $\mathcal{A}_F$ satisfies

(i) $A_{F,g} \in C^2(\bigwedge_s^2 T\mathbb{R}^n \setminus \{0\})$,

(ii) $A_{F,g}$ is uniformly elliptic

(iii) and $\sup_{x\in\mathbb{R}^n} \left\| A_{F,g}(x,\cdot) - |\cdot|_{g_x, \bigwedge^2 T_x\mathbb{R}^n} \right\|_{C^2\left(\mathbb{S}_{g_x}^{\binom{n}{2}-1}\right)} < \frac{1}{5}$.

Compare condition (iii) to the smallness condition on

$$\sup_{x\in\mathbb{R}^n} \left\| F(x,\cdot) - |\cdot|_{g_x} \right\|_{C^2(\mathbb{S}_{g_x}^{n-1})}$$

in the main theorem.

These three goals can be achieved by using functional analytic properties of the so-called ***Radon transform***.

Motivation: An easy calculation

Recall from earlier for $\sigma \in \bigwedge_s^m T_x N$:

$$A_{F,g}(x,\sigma) = \frac{\Omega_m}{\mathcal{H}^m_{g_x}\big(B_{F_x} \cap \langle\sigma\rangle\big)} |\sigma|_{g_x, \bigwedge^m T_x N}.$$

Using spherical coordinates in the Hilbert space $(\langle\sigma\rangle, g_x|_{\langle\sigma\rangle})$:

$$\mathcal{H}^m_{g_x}(B_{F_x} \cap \langle\sigma\rangle) = \int_{\mathbb{S}^{m-1}_{g_x|_{\langle\sigma\rangle}}} \int_0^\infty \chi_{B_{F_x} \cap \langle\sigma\rangle}(s\theta) \;\; s^{m-1} \mathrm{d}s \; \mathrm{d}\mathcal{H}^{m-1}_{g_x}(\theta).$$

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$$\frac{\mathcal{H}^m_{g_x}\left(B_{F_x} \cap \langle\sigma\rangle\right)}{\Omega_m} = \frac{1}{m\Omega_m} \int_{\mathbb{S}^{m-1}_{g_x|_{\langle\sigma\rangle}}} F(x,\cdot)^{-m} \mathrm{d}\mathcal{H}^{m-1}_{g_x}$$

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$(\langle\sigma\rangle, g_x|_{\langle\sigma\rangle}) \hookrightarrow (T_xM, g_x)$ isometrically

$$= \frac{1}{\omega_{m-1}} \int_{\mathbb{S}^{n-1}_{g_x} \cap \langle\sigma\rangle} F(x,\cdot)^{-m} \mathrm{d}\mathcal{H}^{m-1}_{g_x}.$$

The Radon transform: Definition

Let $\mathrm{Gr}_m(\mathbb{R}^n)$ denote the Grassmannian, the set of m-dimensional subspaces of $\mathbb{R}^n$.

The linear operator $\mathcal{R}\colon C(\mathbb{S}^{n-1}) \to C(\mathrm{Gr}_m(\mathbb{R}^n))$ defined by

$$\mathcal{R}[f](E) = \frac{1}{\omega_{m-1}} \int_{\mathbb{S}^{n-1}\cap E} f \, \mathrm{d}\mathcal{H}^{m-1}$$

is called the ***Radon transform*** on $\mathbb{R}^n$.

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is called the ***Radon transform*** on $\mathbb{R}^n$.
This can be lifted to a linear operator $\mathcal{R}_g$ on the sphere bundle and the Grassmann bundle of (N, g).

Corollary

The area integrand $A_{F,g}$ satisfies

$$A_{F,g}(x,\sigma) = \left(\mathcal{R}_g[F(x,\cdot)^{-m}](\langle\sigma\rangle)\right)^{-1} |\sigma|_{g_x, \wedge^m T_x N}.$$

The Radon transform: Properties

- Downside to using $A_{F,g}$:
  In high codimension the base space $\bigwedge_s^m TN$ is a bundle of cones and difficult to handle.
- Benefits of the Radon transform:
  Base spaces $\mathbb{S}^{n-1}$ and $\mathrm{Gr}_m(\mathbb{R}^n)$ are homogeneous $O(n)$-spaces with a lot of algebraic and geometric structure.
- By the Plücker embedding both base spaces are related via

$$\mathrm{Gr}_m(\mathbb{R}^n) \cong (\mathbb{S}(\textstyle\bigwedge^m \mathbb{R}^n) \cap \bigwedge_s^m(\mathbb{R}^n))/\mathbb{Z}_2.$$

The Radon transform: Properties

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Lemma ($O(n)$-Equivariance)

For any $f \in C(\mathbb{S}^{n-1})$ and $Q \in O(n)$:

$$\mathcal{R}[f] \circ Q = \mathcal{R}[f \circ Q].$$

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Taking derivative at $Q = \mathrm{Id}_{\mathbb{R}^n}$ leads to:

Theorem (Differentiability)
Suppose $f \in C^1(\mathbb{S}^{n-1})$. Then $\mathcal{R}[f] \in C^1(\mathrm{Gr}_m(\mathbb{R}^n))$ and for all $X \in \mathfrak{o}(n)$:

$$\mathrm{d}(\mathcal{R}[f])(\mathcal{K}_X) = \mathcal{R}[\mathrm{d}f(\mathcal{K}_X)].$$

↑ ↑ Fundamental vector fields generated by $O(n)$-actions

The Radon transform: Properties

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Theorem (Higher differentiability)
For any $k \geq 0$ the Radon transform

$$\mathcal{R}\colon C^k(\mathbb{S}^{n-1}) \to C^k(\mathrm{Gr}_m(\mathbb{R}^n))$$

is a continuous operator with explicit bounds for its operator norm.

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Theorem (Invertibility, [Helgason, '90])
The Radon transform is an invertible[†] operator when restricted to even functions on the sphere.

[†]Helgason gives an explicit inversion formula

The Radon transform: Summary

- We needed to prove three properties for the area integrand $A_{F,g}$ to apply the theorems of Hildebrandt/von der Mosel:
  (i) $A_{F,g} \in C^2(\bigwedge_s^2 T\mathbb{R}^n \setminus \{0\})$,
  ➔ Use higher differentiability of $\mathcal{R}$
  (ii) $A_{F,g}$ is uniformly elliptic
  ➔ Use invertibility of $\mathcal{R}$
  (iii) and $\sup_{x\in\mathbb{R}^n} \left\| A_{F,g}(x,\cdot) - |\cdot|_{g_x, \wedge^2 T_x\mathbb{R}^n} \right\|_{C^2\left(\mathbb{S}_{g_x}^{\binom{n}{2}-1}\right)} < \frac{1}{5}$.
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  ➔ Use operator norm bounds for $\mathcal{R}$
- The main difficulty in high codimension is that $\bigwedge_s^m TN$ is not a **vector** bundle!
- Use the Radon transform and exploit the nice algebraic structure of the base spaces!

Theorem [P ’20+ε]

Let $M = D$, F a Finsler metric on $N = \mathbb{R}^n$ and g a Riemannian metric. There is a constant $c = c(g, n) > 0$ s.t.

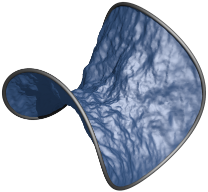
- If

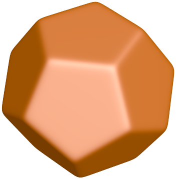
$$\sup_{x \in \mathbb{R}^n} \left\| F(x, \cdot) - |\cdot|_{g_x} \right\|_{C^2(\mathbb{S}^{n-1}_{g_x})} < c,$$

then every g-conformal minimiser f of $\mathcal{A}_F$ in $\mathcal{S}(\Gamma)$ is of class

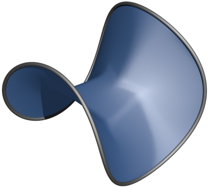
$$W^{2,2}_{\text{loc}}(D; \mathbb{R}^n) \cap C^{1,\mu}(D; \mathbb{R}^n)$$

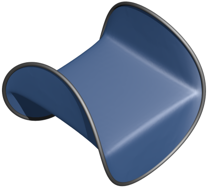
for some $\mu \in (0, 1)$.

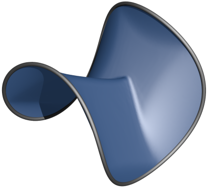




Thank you for your attention!







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