The Plateau problem for the Busemann–Hausdorff area in arbitrary codimension

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Research goal

The classical Plateau problem asks the following questions: Given a simple closed curve $\Gamma \subset \mathbb{R}^3$, is there an embedded or immersed surface M such that $\partial M = \Gamma$ which has least area among all surfaces with the same boundary? How regular is this surface?
The objective is to answer these questions for a weighted area functional (originating in Finsler geometry) and in arbitrary codimension. This generalises the codimension one results in [2].

Background

- A ***Finsler metric*** on a smooth manifold N is a function $F\colon TN \to [0,\infty)$ s.t.:

(F1) ***Regularity***: $F \in C^k(TN \setminus 0) \cap C^0(TN)$, $k \in \{2,3,\dots,+\infty\}$.
(F2) ***Positive homogeneity***: $F(q,tv) = tF(q,v)$ for all $t > 0$, $(q,v) \in TN$.
(F3) ***Ellipticity***: The matrix $g^F_{ij}(q,v) := \left(\frac{1}{2}F^2\right)_{y^i y^j}(q,v)$ is positive definite for any $(q,v) \in TN \setminus 0$.

- A Finsler metric is ***reversible*** if $F(q,v) = F(q,-v)$ for all $(q,v) \in TN$.
- A C^k-immersion $f\colon M^m \to (N^n, F)$ induces a ***pull-back Finsler metric*** $f^\# F$ on M via $(f^\# F)(p,v) := F(f(p), df_p(v))$ for $(p,v) \in TM$.
- Let $g \in \Sigma^2(T'M)$ be a fixed auxiliary Riemannian metric on the domain manifold M. Then the ***Busemann–Hausdorff area*** of $(M, f^\# F)$ is defined as

$$\mathcal{A}_F(f) = \int_M \sigma_{f^\# F, g}\, \mu_g$$

where μ_g is the Riemannian density induced by g and $\sigma_{F,g}\colon M \to \mathbb{R}$ is the function defined by

$$\sigma_{f^\# F,g}(p) = \frac{\mu_g|_p\left(B_1^{g,p}(0)\right)}{\mu_g|_p\left(B_1^{f^\# F,p}(0)\right)}.$$

- The ***m-harmonic symmetrisation*** $F_{(m)}$ is defined as

$$F_{(m)}(q,v) := 2^{\frac{1}{m}}\left((F(q,v))^{-m} + (F(q,-v))^{-m}\right)^{-\frac{1}{m}}$$

for $(q,v) \in TN \backslash 0$. A reversible Finsler metric F coincides with its m-harmonic symmetrisation. However, in general, $F_{(m)}$ is *not* a Finsler metric, in particular, (F3) need not be fulfilled. Therefore, we make the following ***General Assumption*** (GA).

Let F be a Finsler metric on N such that its m-harmonic symmetrisation $F_{(m)}$ is also a Finsler metric on N.

- For a given closed Jordan curve $\Gamma \subset \mathbb{R}^n$ denote by $\mathcal{C}(\Gamma)$ the class of competing admissible disc-type surfaces. A map $f\colon B = B_1(0) \subset \mathbb{R}^2 \to \mathbb{R}^n$ belongs to $\mathcal{C}(\Gamma)$ if $f|_{\partial B}$ is a continuous, weakly monotonic parametrisation of Γ.

Plateau problem for Finsler area – Existence

Theorem ([1, Theorem 1.1])

Let F be a Finsler metric on $N = \mathbb{R}^n$ which satisfies (GA) and fulfils

$$0 < c_F|v|_{\mathbb{R}^n} \le F(p,v) \le C_F|v|_{\mathbb{R}^n} < \infty.$$

For any given rectifiable Jordan curve $\Gamma \subset \mathbb{R}^n$ there exists a surface $f \in \mathcal{C}(\Gamma)$, s.t.

$$\mathcal{A}_F(f) = \inf_{\mathcal{C}(\Gamma)} \mathcal{A}_F(\cdot).$$

In addition, the minimiser f is Euclidean-conformally parametrised on B (w.r.t. auxiliary metric g), i.e. for some smooth function φ:

$$f^\#\langle\cdot,\cdot\rangle_{\mathbb{R}^n} = \varphi g(\cdot,\cdot) \quad \mathcal{H}^2\text{– a.e. on } B.$$

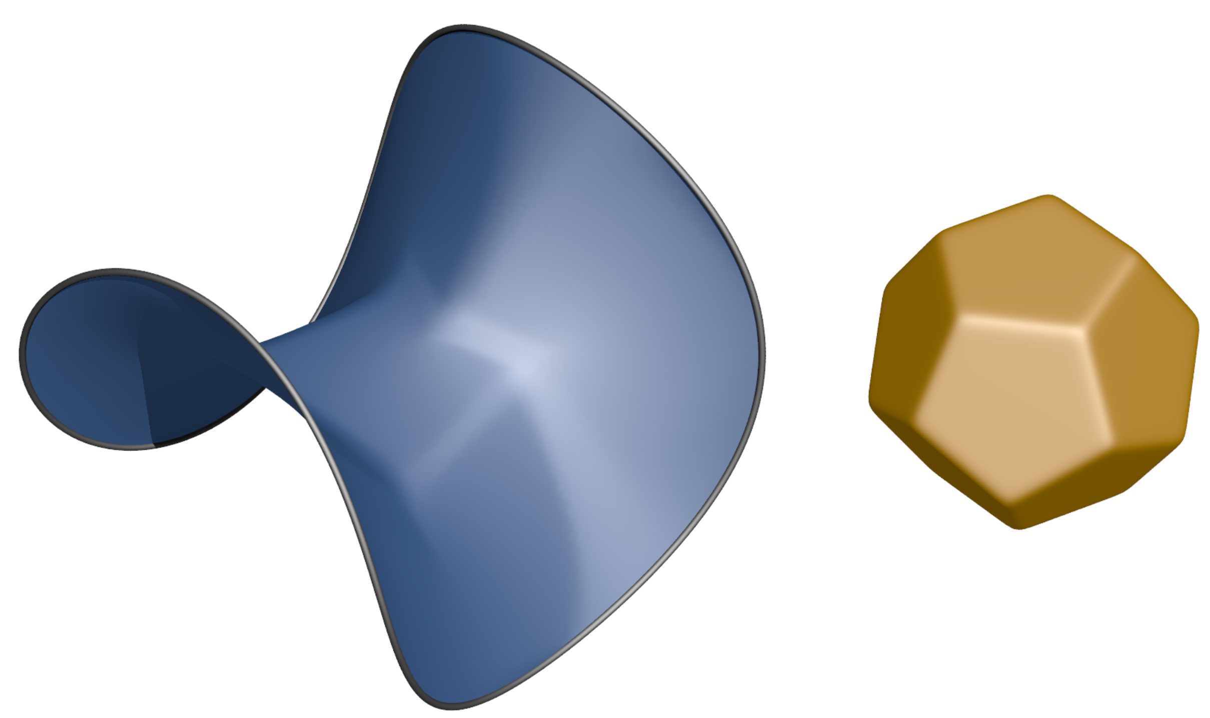
Furthermore, the minimiser f has the following regularity property,

$$f \in C^0(\overline{B};\mathbb{R}^n) \cap C^{0,\alpha}(B;\mathbb{R}^n) \cap W^{1,q}(B,\mathbb{R}^n)$$

for $\alpha := \left(\frac{c_F}{C_F}\right)^2 \in (0,1]$ and some $q > 2$.

Numerical approximation of a minimising surface

This example illustrates that minimisers of anisotropic energy functionals such as $\mathcal{A}_F$ might develop "smooth ridges". In principle, the anisotropy of the F-unit ball favours some directions over others for variations of the surface.



Area minimising surface (left) for a Minkowski-Finsler metric whose unit ball is a smoothened icosahedron (right) – Courtesy of Henrik Schumacher

Idea of proof

- Rewrite the integrand of $\mathcal{A}_F$ as

$$a^F_m\colon \bigwedge\nolimits^m_s TN \to \mathbb{R}_+, (q,\sigma) \mapsto \frac{\mathcal{H}^m(B^m_1(0))}{\mathcal{H}^m\left(B_1^{F,q}(0) \cap \langle\sigma\rangle\right)},$$

where the set $\bigwedge^m_s(TN)$ denotes the bundle of simple tangent m-vectors, s.t.

$$\mathcal{A}_F(f) = \int_M a^F_m\left(f, df(\tfrac{\partial}{\partial u^1}) \wedge \cdots \wedge df(\tfrac{\partial}{\partial u^m})\right) du^1 \wedge \cdots \wedge du^m.$$

- Show that the mapping a^F_m is

(i) ***positively 1-homogeneous*** in its second component,
(ii) ***continuous*** as a function on $\mathbb{R}^n \times \mathbb{R}^{\binom{n}{m}}$ for $N = \mathbb{R}^n$.
(iii) ***positive definite***, i.e. there exist $0 < M_1 \le M_2$ s.t. for $(q,\tau) \in \bigwedge^m_s(TN)$:

$$M_1\|\tau\|_{g,\bigwedge^m(T_qN)} \le a^F_m(q,\tau) \le M_2\|\tau\|_{g,\bigwedge^m(T_qN)}$$

(iv) ***convex*** in its second component for $m = 2$ and reversible F.

- Use the existence and regularity theory developed by Hildebrandt and von der Mosel [3] for the ***Cartan integrand*** a^F_2 if F is reversible.
- Using that $a^{F_{(m)}}_m = a^F_m$ conclude the proof also for irreversible F.

Plateau problem for Finsler area – Regularity

Theorem in codimension one ([2, Theorem 1.4])

There is a universal constant $\delta \in (0,1)$ such that any Euclidean-conformally parametrised (w.r.t the auxiliary metric g) minimiser f of $\mathcal{A}_F$ is of class $W^{2,2}_{loc}(B;\mathbb{R}^3) \cap C^{1,\mu}(B;\mathbb{R}^3)$ for some $\mu \in (0,1)$ provided that the Finsler metric satisfies

$$\sup_{p\in\mathbb{R}^3}\left\|\left|\nabla^2_{\mathbb{S}^2}\left(F(p,\cdot) - |\cdot|_{\mathbb{R}^3}\right)\right|_{\mathbb{R}^3}\right\|_{L^\infty(\mathbb{S}^2)} < \delta,$$

i.e. F and $|\cdot|_{\mathbb{R}^3}$ are comparable up to second order.

An extension of this result to arbitrary codimension is work in progress.

Literature

[1] S. Pistre and H. von der Mosel. The Plateau problem for the Busemann–Hausdorff area in arbitrary codimension. *European Journal of Mathematics* (2017).

[2] P. Overath, H. von der Mosel. Plateau's problem in Finsler 3-space. *Manuscripta Mathematica* (2014).

[3] S. Hildebrandt, H. von der Mosel. On two-dimensional parametric variational problems. *Calculus of Variations and Partial Differential Equations* (1999).

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