„5 vor 12“ - Die RWTH Wissenschaftsnacht

Eine Veranstaltung der RWTH

The picture-hanging problem

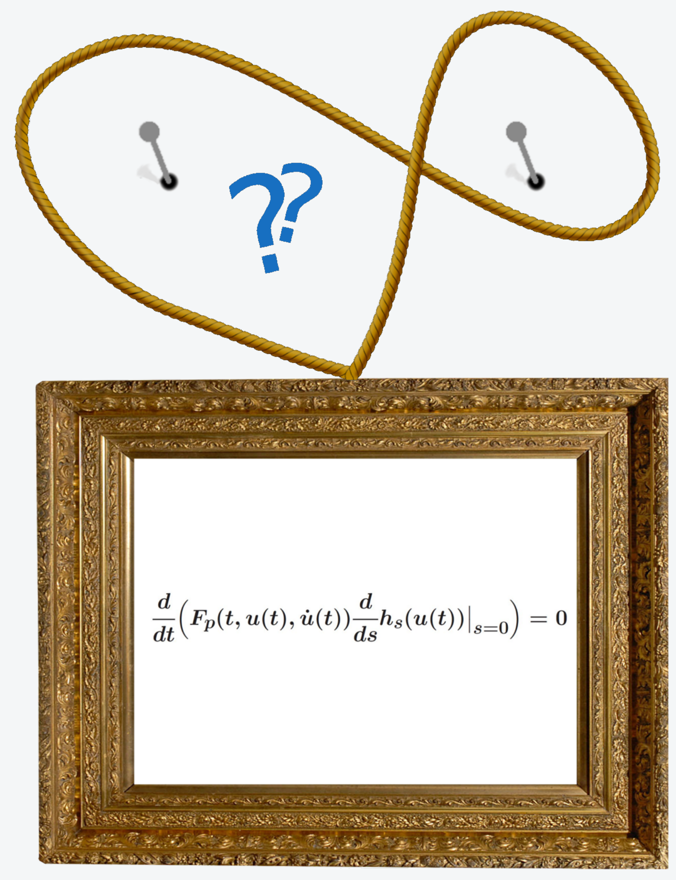
Sven Pistre, Elisabeth Wacker

RWTH AACHEN UNIVERSITY

Problem

There was a painting in Emmy Noether's office in Princeton. Not only was it special for depicting her famous theorem about symmetries and conserved quantities but more interestingly for the way it was installed on the wall. Emmy Noether had connected a thread to the frame and hung up the painting on two nails in the wall. Whenever she was visited by students during office hours she told them she had wound the thread around the nails in such a way that the painting would fall down if any ***one*** of the two nails (***no matter which***) were to be pulled out of the wall. The students were astonished and thought:

- How exactly had Emmy Noether accomplished this?
- Could she have achieved this with three nails as well?
- Would a constellation of 100000 nails also be possible?

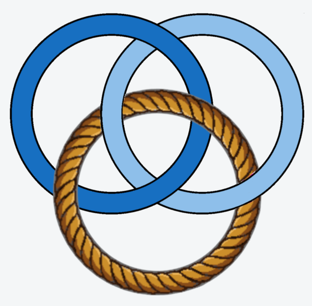


This is of course a fictitious anecdote, however, Emmy Noether (* 23. March 1882, † 14. April 1935) was the greatest female mathematician of her time with many far-reaching contributions to mathematical physics, the calculus of variations and algebra.

Connections to knot theory

Borromaen rings

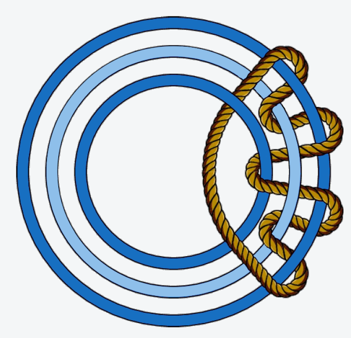
The Borromean rings are three rings which are inseparable but such that no two of the rings are themselves linked. Such a constellation corresponds to a solution of the 2-nail-problem (and vice versa):

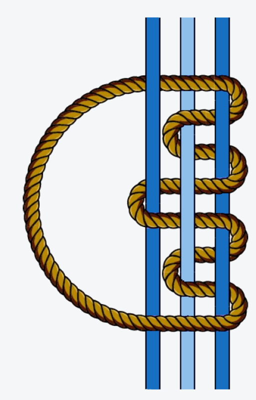


A ring can be regarded as a straight line by “cutting it open” and extending it indefinitely. Mathematically this is the stereographic projection of $\mathbb{S}^1$ onto $\mathbb{R} \cup \{\infty\}$. If one identifies two of the Borromean rings with such a straight line (“nail”) then the remaining ring corresponds exactly to the thread.

Brunnian links

A solution to the generalised picture-hanging problem with n nails can be given by a similar construction from knot theory. A constellation of $n+1$ inseparable rings such that the removal of any one ring yields a constellation of n rings which are unlinked is called a Brunnian link. If one “cuts open” n rings in a Brunnian link the remaining ring corresponds to a constellation of the thread which solves the n-nail problem.





Connections to algebraic topology

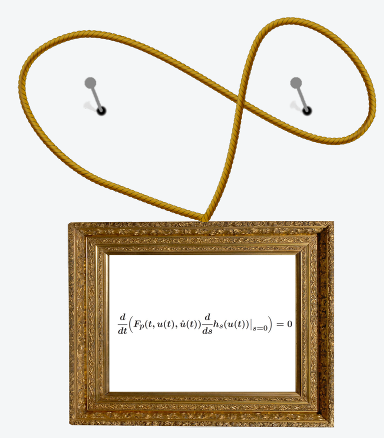
To construct an explicit solution it is helpful to consider the problem on a more abstract level: By fixing one point (picture frame) in the two-dimensional plane (wall) one can consider all possible closed loops (thread) which start at this one fixed point. Mathematically these loops form the fundamental group of the plane. If there are no nails in the wall then there is no elementary (meaning: non-constant) loop. If there is one nail in the wall then there is (up to orientation) one elementary loop; if there are two nails then there are two elementary loops, etc (see figure on the right). Any possible solution can be assembled by these elementary loops. Mathematically one computes the fundamental group of the plane with n points (nails) removed. This is the free group over n generators. Introduce a symbol a_i for any of the n nails where

- a_i^+: “Guide the thread over nail i **clockwise**.”
- a_i^-: “Guide the thread over nail i **anti-clockwise**.”

We are looking for a “word” that can be built from the letters $a_1^\pm, a_2^\pm \ldots, a_n^\pm$, such that the removal of any one letter a_i already “cancels” all the remaining letters. Here we always have the rules that $a_j^+ a_j^- = 1$ (the “empty” word) and $a_j^n a_j^m = a_j^{n+m}$, but $a_j^\pm a_k^\pm \neq a_k^\pm a_j^\pm$ whenever $k \neq j$. For example, $a_1^+ a_2^- a_3^+ a_3^- a_2^+ = a_1^+$.

Can you construct such a word consisting of two letters $a_1^\pm, a_2^\pm$? If so, is it possible to use this word to create another word consisting of three letters $a_1^\pm, a_2^\pm, a_3^\pm$? Is this an efficient way to solve the n-nail problem?

Two elementary loops



This loop corresponds to $a_1^+ a_2^-$

More infos?