The Plateau problem for the Busemann–Hausdorff area in arbitrary codimension

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Introduction

In the classical Plateau problem one seeks minimal surfaces spanning a prescribed closed boundary contour $\Gamma \subset \mathbb{R}^3$. This can be achieved, e.g. by minimising the Euclidean area functional in a suitable class of parametric surfaces; see [4]. We investigate the Plateau problem in $\mathbb{R}^n$ endowed with a Finsler structure and a particular notion of area.

Let $\mathcal{N} = \mathcal{N}^n$ be a smooth n-dimensional manifold and $T\mathcal{N}$ its tangent bundle. A non-negative function $F\colon T\mathcal{N} \to [0,\infty)$ is a ***Finsler metric*** if the following three conditions are satisfied:

(F1) ***Regularity*** : $F \in C^k(T\mathcal{N}\setminus o) \cap C^0(T\mathcal{N})$, $k \in \mathbb{N} \cup \{+\infty\}, k \geq 2$.

(F2) ***Positive homogeneity***: $F(q,tv) = tF(q,v)$ for all $t > 0$ and $(q,v) \in T\mathcal{N}$.

(F3) ***Ellipticity***: The matrix $g^F_{ij}(q,v) := \left(\frac{1}{2}F^2\right)_{y^iy^j}(q,v)$ is positive definite for any $(q,v) \in T\mathcal{N}\setminus o$.

The pair $(\mathcal{N},F)$ is a ***Finsler manifold***. A Finsler metric is ***reversible*** if $F(q,v) = F(q,-v)$ for all $(q,v) \in T\mathcal{N}$. Any C^k-immersion $X : \mathcal{M}^m \to (\mathcal{N}^n, F)$ for $0 < m \leq n$ and $k \geq 2$ induces a ***pull-back Finsler metric*** X^*F on $\mathcal{M}$ via

$$X^*F(p,v) := F(X(p), dX_p(v))$$

for $(p,v) \in T\mathcal{M}$. The ***m-dimensional Busemann–Hausdorff volume form on*** $\mathcal{M}$ (w.r.t. the Finsler metric X^*F on $\mathcal{M}$) is defined in local coordinates as $dV_{X^*F}(p) := \sigma_{X^*F}(p)du^1 \wedge du^2 \wedge \cdots \wedge du^m$ where

$$\sigma_{X^*F}(p) := \frac{\mathscr{H}^m(B^m_1(0))}{\mathscr{H}^m\left(\left\{v \in \mathbb{R}^m :\ F\left(X(p), v^\alpha dX_p\left(\frac{\partial}{\partial u^\alpha}\big|_p\right)\right) < 1\right\}\right)}. \tag{1}$$

The ***Busemann–Hausdorff area of the immersion*** X is then given by $\mathrm{area}^F_{\mathcal{M}}(X) := \int_{\mathcal{M}} dV_{X^*F}$.

Further, consider the ***m-harmonic symmetrisation*** $F_{(m)}$ defined as the reversible function

$$F_{(m)}(q,v) := 2^{\frac{1}{m}}\left((F(q,v))^{-m} + (F(q,-v))^{-m}\right)^{-\frac{1}{m}}$$

for $(q,v) \in T\mathcal{N}\setminus o$. A reversible Finsler metric F coincides with its m-harmonic symmetrisation. However, in general, $F_{(m)}$ is *not* a Finsler metric, in particular, (F3) need not be fulfilled. Therefore, we make the following ***General Assumption***.

Let F be a Finsler metric on an n-dimensional smooth manifold $\mathcal{N}$ such that its m-harmonic symmetrisation $F_{(m)}$ is also a Finsler metric on $\mathcal{N}$. (GA)

Main result

In order to state the existence result for the Finsler-Plateau problem for two-dimensional surfaces in any codimension we define for a given closed Jordan curve $\Gamma \subset \mathbb{R}^n$ the class of admissible surfaces

$$\mathscr{C}(\Gamma) := \left\{X \in W^{1,2}(B,\mathbb{R}^n) :\ X\big|_{\partial B} \text{ is a continuous and weakly monotonic parametrisation of } \Gamma\right\}.$$

Theorem (Plateau problem for Busemann–Hausdorff area)**.**
Let F be a Finsler metric on $\mathcal{N} = \mathbb{R}^n$ which satisfies (GA) *for $m = 2$ and assume in addition that*

$$0 < c_F := \inf_{\mathbb{R}^n \times \mathbb{S}^{n-1}} F(\cdot,\cdot) \leq \sup_{\mathbb{R}^n \times \mathbb{S}^{n-1}} F(\cdot,\cdot) = \|F\|_{L^\infty(\mathbb{R}^n\times\mathbb{S}^{n-1})} =: C_F < \infty. \tag{D$_F$}$$

Then for any given rectifiable Jordan curve $\Gamma \subset \mathbb{R}^n$ there exists a surface $X \in \mathscr{C}(\Gamma)$, such that

$$\mathrm{area}^F_B(X) = \inf_{\mathscr{C}(\Gamma)} \mathrm{area}^F_B(\cdot).$$

In addition, the minimiser X is Euclidean conformally parametrised on B, i.e.

$$\left|\frac{\partial X}{\partial u^1}\right|^2 = \left|\frac{\partial X}{\partial u^2}\right|^2 \quad \text{and} \quad \left\langle \frac{\partial X}{\partial u^1}, \frac{\partial X}{\partial u^2}\right\rangle_{\mathbb{R}^2} = 0 \quad \mathscr{H}^2\text{– a.e. on } B. \tag{CONF}$$

Furthermore, the minimiser X has the following regularity property,

$$X \in C^0\left(\overline{B},\mathbb{R}^n\right) \cap C^{0,\alpha}\left(B,\mathbb{R}^n\right) \cap W^{1,q}\left(B,\mathbb{R}^n\right) \tag{R}$$

for $\alpha := \left(\frac{c_F}{C_F}\right)^2 \in (0,1]$ and some $q > 2$.

The existence and regularity theory for the Plateau problem for a general class of ***Cartan functionals*** was developed by Hildebrandt and von der Mosel in a series of papers, see e.g. [5, 6]. We reformulate the Busemann–Hausdorff area functional and use the following general result to prove the main theorem above.

Theorem (Plateau problem for Cartan integrands, [5])**.**
Suppose $I \in C^0\left(\mathbb{R}^n \times \mathbb{R}^{\binom{n}{2}};\mathbb{R}\right)$ is positively 1-homogeneous, convex in its second component and satisfies (D) *below. Then for any given rectifiable Jordan curve $\Gamma \subset \mathbb{R}^n$ there exists a surface $X \in \mathscr{C}(\Gamma)$, s. t. $\mathscr{I}(X) = \inf_{\mathscr{C}(\Gamma)} \mathscr{I}(\cdot)$ where $\mathscr{I}(X) := \int_B I(X, X_{u^1} \wedge X_{u^2})\,du^1 \wedge du^2$.*
In addition, the minimiser X satisfies condition (CONF) *and fulfils property* (R) *for $\alpha := \frac{M_1}{M_2} \in (0,1]$ and some $q > 2$.*

Idea of the proof

Rewrite the integrand (1) of the Busemann–Hausdorff area functional in terms of the mapping

$$a^F_m\colon \bigsqcup_{q\in\mathcal{N}} \textstyle\bigwedge^m_s(T_q\mathcal{N}) \to \mathbb{R}_+, (q, w_1 \wedge w_2 \wedge \cdots \wedge w_m) \mapsto \dfrac{\mathscr{H}^m(B^m_1(0))}{\mathscr{H}^m\left(\{v \in \mathbb{R}^m :\ F(q, v^\alpha w_\alpha) < 1\}\right)},$$

where the set $\bigwedge^m_s(T_q\mathcal{N})$ denotes the set of simple tangent m-vectors, s.t. the area functional takes the form

$$\mathrm{area}^F_{\mathcal{M}}(X) = \int_{\mathcal{M}} a^F_m\left(X, dX\left(\tfrac{\partial}{\partial u^1}\right) \wedge \cdots \wedge dX\left(\tfrac{\partial}{\partial u^m}\right)\right) du^1 \wedge \cdots \wedge du^m.$$

We show that the mapping a^F_m is

- ***positively*** (in fact, absolutely) ***1-homogeneous*** in its second component,
- ***continuous*** as a function on $\mathbb{R}^n \times \mathbb{R}^{\binom{n}{m}}$ for $\mathcal{N} = \mathbb{R}^n$.

These properties identify a^F_m as a ***Cartan integrand***. All existence results for minimisers of Cartan functionals require two additional conditions on the integrand.

- Firstly, a^F_m needs to be ***positive definite***, i.e. there exist $0 < M_1 \leq M_2$ s.t.

$$M_1\|\tau\|_{g,\bigwedge^m(T_q\mathcal{N})} \leq a^F_m(q,\tau) \leq M_2\|\tau\|_{g,\bigwedge^m(T_q\mathcal{N})} \quad \text{for all } (q,\tau) \in \textstyle\bigwedge^m_s(T\mathcal{N}), \tag{D}$$

  where g is an arbitrary auxiliary Riemannian metric on $\mathcal{M}$.

Due to Overath [7] we have that for any $(q, w_1 \wedge \ldots \wedge w_m) \in \bigwedge^m_s(T\mathcal{N})$

$$a^F_m(q, w_1 \wedge w_2 \wedge \cdots \wedge w_m) = \left(\frac{1}{\mathscr{H}^{m-1}(\mathbb{S}^{m-1})}\int_{\mathbb{S}^{m-1}} \frac{1}{(F(q, \theta^\alpha w_\alpha))^m}\, d\mathscr{H}^{m-1}(\theta)\right)^{-1}.$$

This can be used to transfer the L^∞-bounds (D$_F$) on F to condition (D) for a^F_m by choosing $M_i := c^m_i$ for $i = 1,2$. Additionally, this shows that

$$\mathrm{area}^F_{\mathcal{M}}(X) = \mathrm{area}^{F_{(m)}}_{\mathcal{M}}(X). \tag{2}$$

- Secondly, a^F_m needs to be ***convex*** in its second component.

In codimension one and for reversible Finsler metrics, this is a classical result from convex analysis by Busemann.

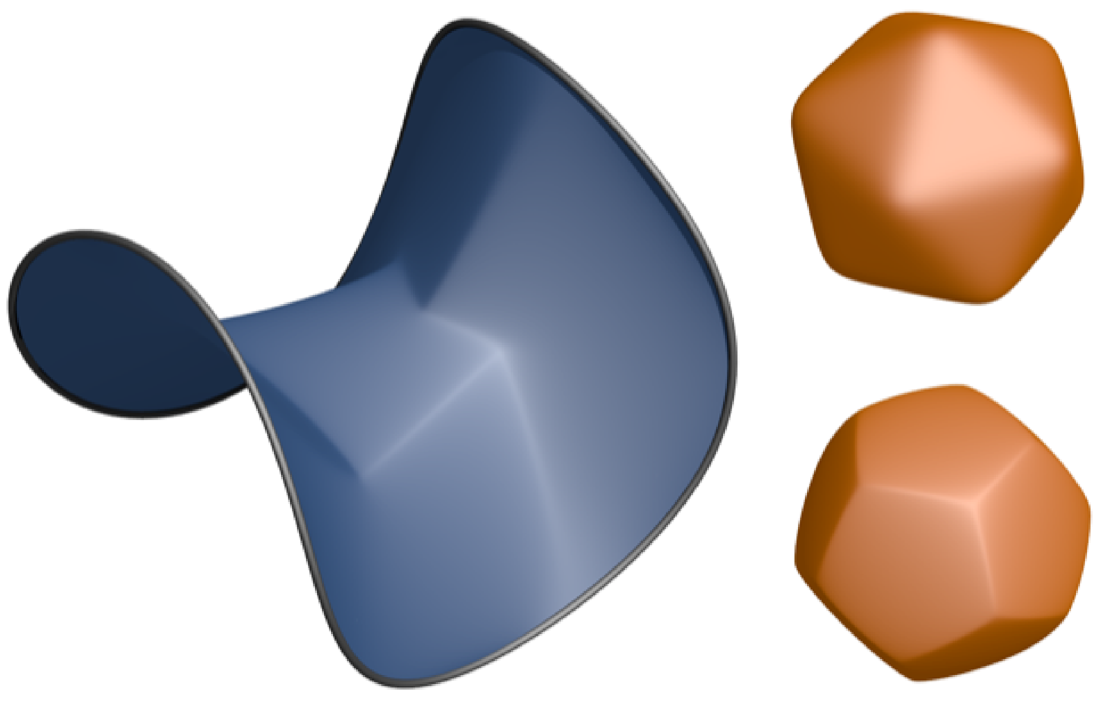
More recently, Burago and Ivanov [3] have shown for $m = 2$ and reversible F

- that there exists ***convex extension*** of a^F_2 to the vector bundle $\bigwedge^2(T\mathcal{N})$ in any codimension,

which inherits the ***absolute homogeneity*** and ***continuity***.

A different notion of volume introduced by Bernig [2] also contains this result as a special case.

Finally, we use the existence result for Cartan functionals to solve the Plateau problem for the Busemann–Hausdorff area for reversible F and then use (2) to also show existence for irreversible Finsler metrics F.



Area minimising surface w.r.t. a Finsler metric whose unit ball is a smoothened icosahedron.
(Courtesy of Henrik Schumacher)

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S. Pistre and H. von der Mosel. The Plateau problem for the Busemann–Hausdorff area in arbitrary codimension. *European Journal of Mathematics*, 3(4):953-973, Dec 2017.

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